The initial value problem for entropic systems of conservation laws treated as a concave maximization problem of optimal transport type

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en collaboration avec le projet CNRS-INRIA MOKAPLAN

"Waves diffracted by PATRICK JOLY",
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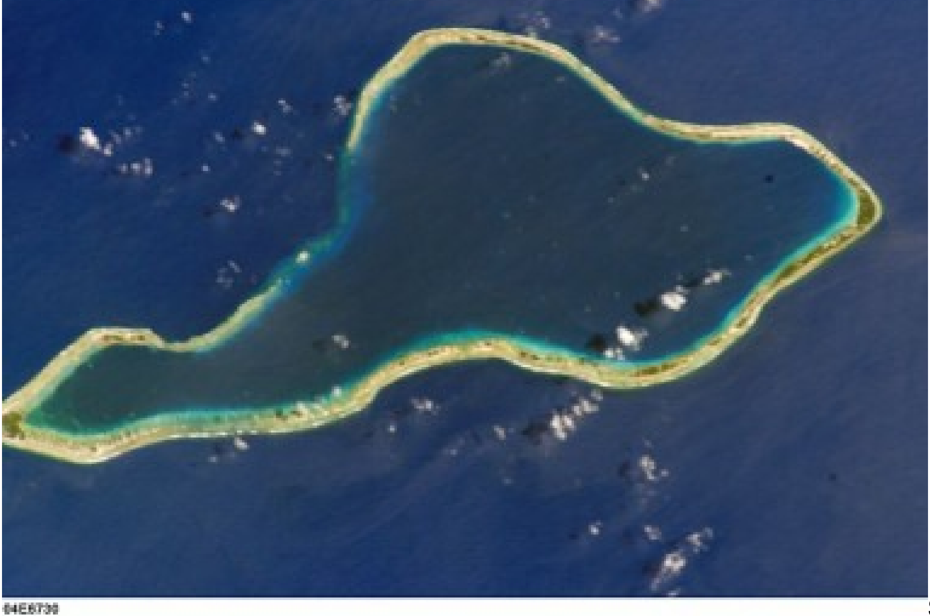

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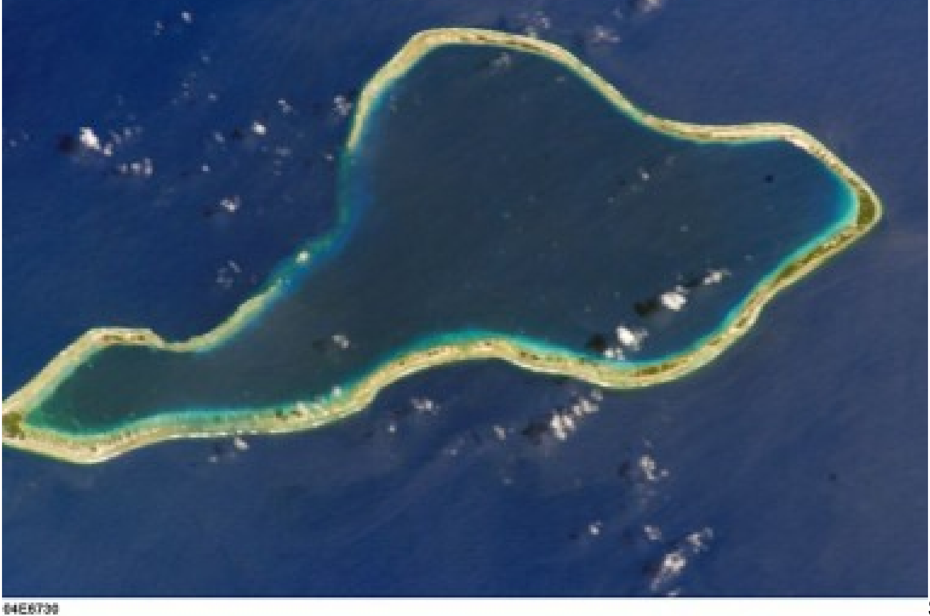

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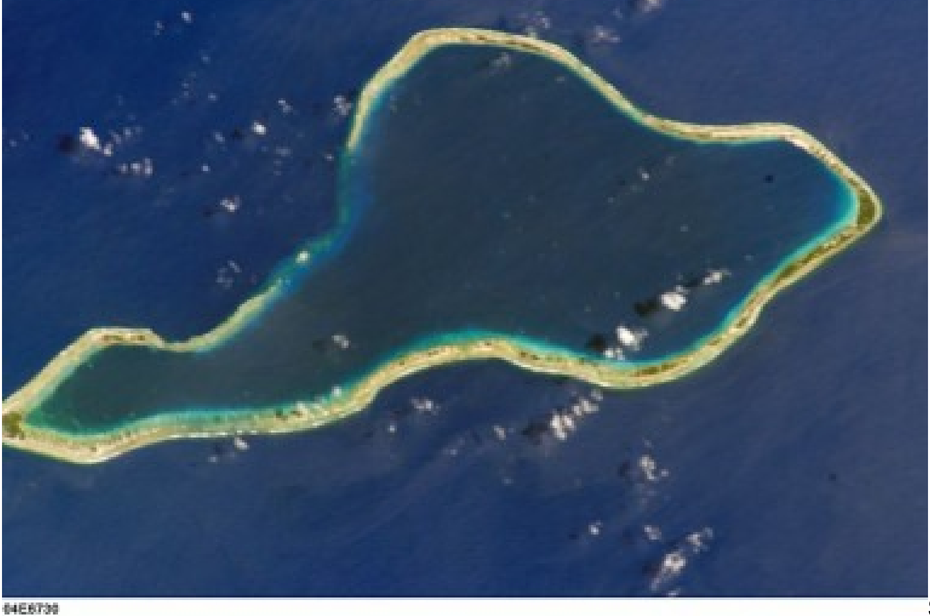

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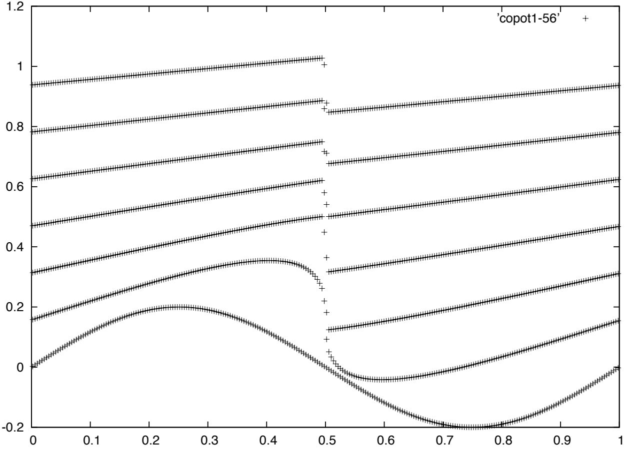
Typically, these systems are locally well-posed, with possible formation of shock waves.



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formation of a shock wave ($x \in \mathbb{R}/\mathbb{Z}$, $t \in [0, 1]$), shown at 7 successive times, with the Engquist-Osher scheme (256 grid points, CFL=0.25).

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Observe that the problem is not trivial since usually weak solutions are not unique and do not preserve the entropy during their evolution.

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$$\inf_U \sup_A \int_{[0,T]\times\mathbb{T}^d} \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U)$$

$$-\int_{\mathbb{T}^d} A(0,\cdot)\cdot U_0$$

where $A = A(t,x) \in \mathbb{R}^m$ is smooth with $A(T,\cdot) = 0$. Here U_0 is the initial condition and T the final time.

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0, all test functions A acting like Lagrange multipliers.

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leads to a *concave* maximization problem in A, namely

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$$= \sup_A \int_{[0,T]\times\mathbb{T}^d} -K(\partial_t A, \nabla A) - \int_{\mathbb{T}^d} A(0,\cdot)\cdot U_0$$

$$K(E,B) = \sup_{U\in\mathbb{R}^m} E\cdot U + B\cdot F(U) - \mathcal{E}(U),\ (E,B)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}$$

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N.B. Since K is automatically convex, this leads to a (possibly degenerate)

space-time elliptic system in A, which can be solved by rather standard methods.

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(*) more precisely if, $\forall (t,x)$, $\mathcal{E}" - (T-t)F" \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$, which implies

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...But what about shocks and large T???

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$$= \sup_A \int_{[0,T]\times\mathbb{T}} -\frac{(\partial_t A)^2}{2(1-\partial_x A)} - \int_{\mathbb{T}} A(0,\cdot)u_0.$$

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Introducing $\rho = 1 - \partial_x A \geq 0$, $q = \partial_t A \in \mathbb{R}$, this problem is equivalent to

$$-\inf_{\rho,q} \int_{[0,T]\times\mathbb{T}} \frac{q^2}{2\rho} - qu_0$$

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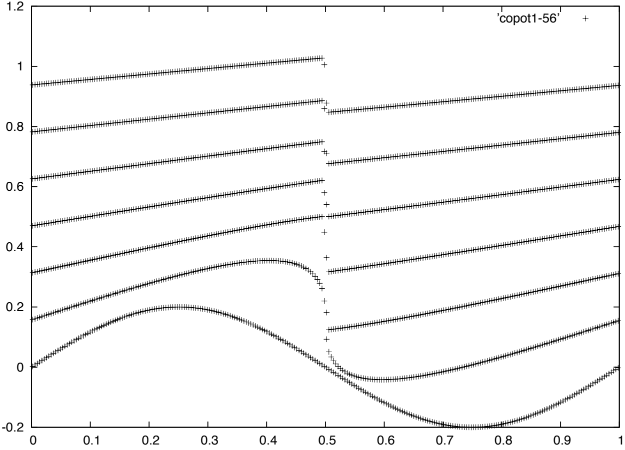
where (ρ, q) are subject to $\partial_t \rho + \partial_x q = 0, \ \ \rho(T,\cdot) = 1$. This is very close to an optimal transport problem with quadratic cost, in its "Benamou-Brenier" formulation.



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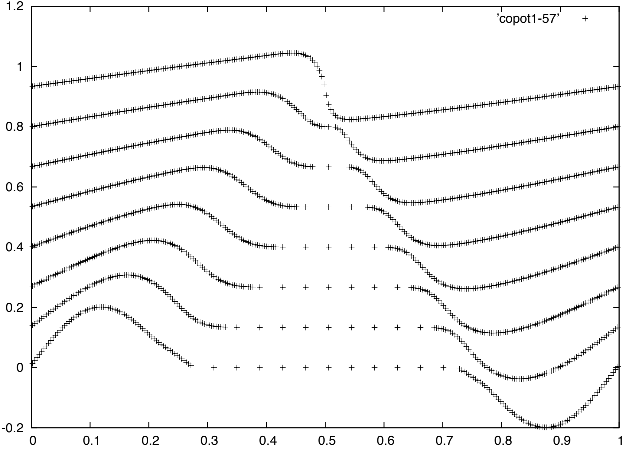
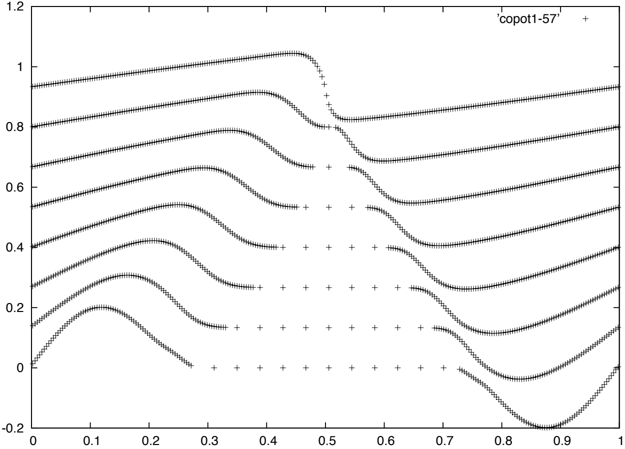


Recovery of the entropy solution at time $t = 1$ by convex minimization.



Recovery of the entropy solution at time $t = 1$ by convex minimization.
Notice that the entropy solution is (approximately!) recovered *only* at final time $T = 1$. Indeed the shock formation has been automatically postponed to $t = 1$.

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and can be interpreted, with its dual form, as a "mean-field game" à la Lasry-Lions:

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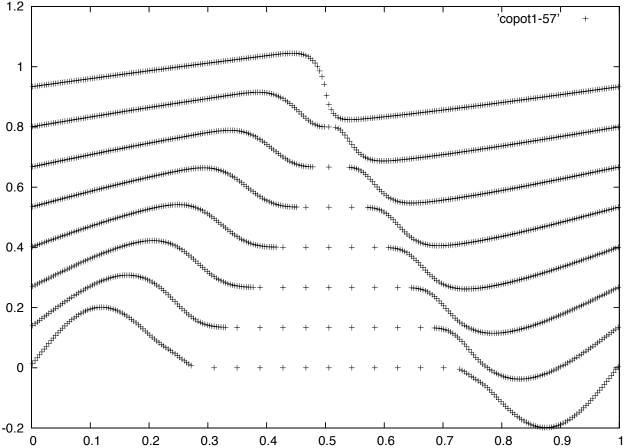
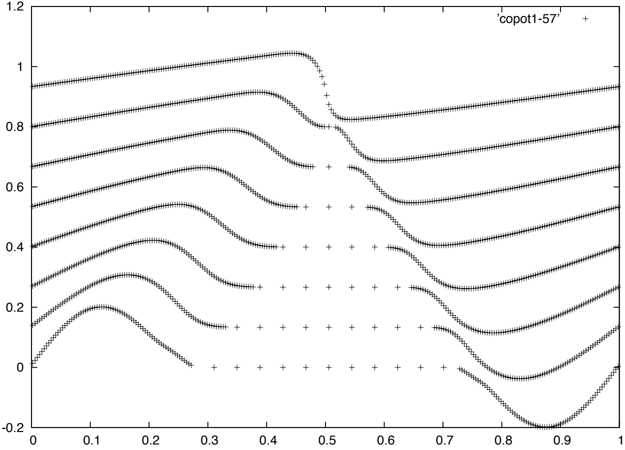


Recovery of the entropy solution at time $t = 1$ by convex minimization.



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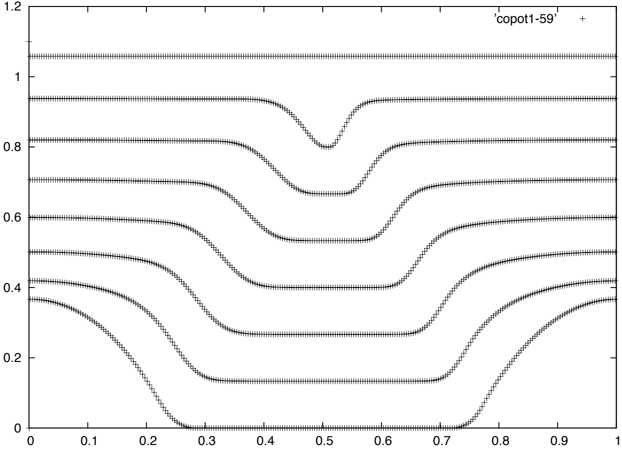
Notice the vacuum area before the shock forms exactly at the final time $T = 1$.



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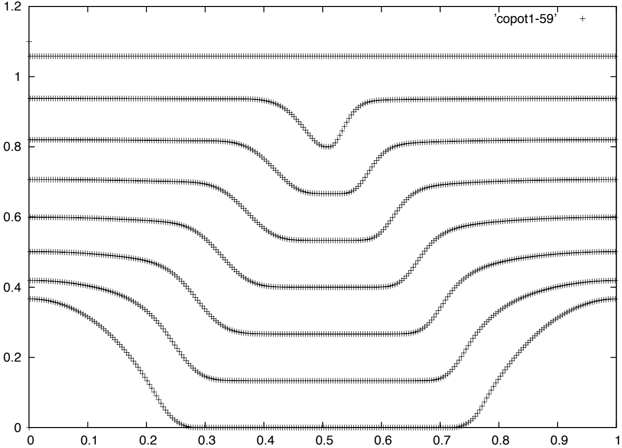

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CHER PATRICK, BIENVENUE AU CLUB 60!!!