Density and trace results in generalized fractal networks

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Outline

Main questions

If $\mathcal{T}$ is an infinite *p*-adic tree and μ a weight function, we are interested in the two following questions:

1. Find NSC such that $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$?

$\mathcal{H}^1_{\mu,0}(\mathcal{T})$ being the closure in $\mathcal{H}^1_\mu(\mathcal{T})$ of compactly supported functions.

2. If $\mathcal{H}^1_\mu(\mathcal{T}) \neq \mathcal{H}^1_{\mu,0}(\mathcal{T})$, define a trace space (at infinity) of elements of $\mathcal{H}^1_\mu(\mathcal{T})$.

For some particular trees and weights in the finite difference version, see

B. Maury, D. Salort, and C. Vannier.
Trace theorems for trees, application to the human lungs.
Network and Heteregeneous Media, 4(3):469 – 500, 2009.

p-adic trees

Given p in $\mathbb{N}^*$, we denote the following set of indexes in $\mathbb{N}^2$:

$$\begin{aligned}
\mathbb{E}_p &= \left\{(\ell, j) \in \mathbb{N}^2 \text{ such that } 0 \leq j \leq p^\ell - 1\right\}, \\
\mathbb{V}_p &= (0,0) \cup \left\{(\ell, j) \in \mathbb{N}^2 \text{ such that } \ell \geq 1 \text{ and } 0 \leq j \leq p^{\ell-1} - 1\right\}.
\end{aligned}$$

Definition

$\mathcal{T}$ is a p-adic tree if there exists two families $\mathcal{E} = (e_{\ell,j})_{(\ell,j)\in\mathbb{E}_p}$ (set of edges) and $\mathcal{V} = (v_{\ell,j})_{(\ell,j)\in\mathbb{V}_p}$ (set of nodes) such that:

- *each $v_{\ell,j}$ is a point of $\mathbb{R}^d$,*
- *each $e_{\ell,j}$ is a straight segment in $\mathbb{R}^d$ of length $L_{\ell,j}$, whose extremities are $v_{\ell,\lfloor j/p \rfloor}$ and $v_{\ell+1,j}$,*
- $(\ell, j) \neq (\ell', j') \Rightarrow v_{\ell,j} \neq v_{\ell',j'}$,
- $(\ell, j) \neq (\ell', j') \Rightarrow e_{\ell,j} \cap e_{\ell',j'} = \emptyset$.

An example

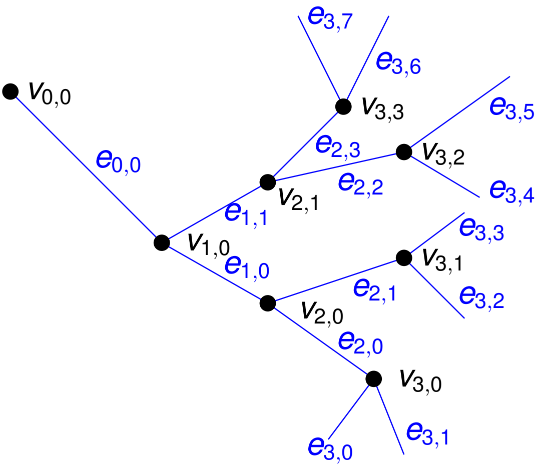

Figure: A dyadic tree. We circle nodes and we color edges in blue.

Subtrees

$\mathcal{T}^\ell$ = subtree of $\mathcal{T}$ made of the edges up to the ℓ-th generation.

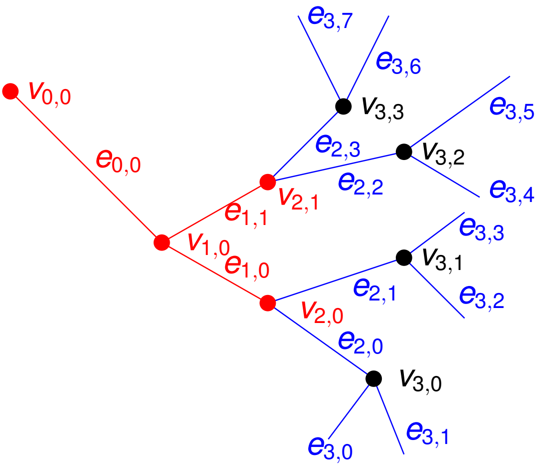

Figure: The dyadic subtree $\mathcal{T}^1$ in red.

Weight on a p-adic tree

Definition

Let us consider a p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ and a function $\mu : \mathbb{E}_p \to \mathbb{R}$. One says that μ is a weight on $\mathcal{T}$ if and only if

$$0 < \mu_{\ell,j} := \mu(\ell, j) < \infty, \quad \forall (\ell, j) \in \mathbb{E}_p.$$

In this case, we denote the weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$.

By abuse of notation, we also denote by μ the function from $\mathcal{E}$ to $\mathbb{R}$ defined by

$$\mu(\mathbf{x}) = \mu_{\ell,j}, \forall \mathbf{x} \in e_{\ell,j}.$$

Weighted L^2 spaces

Definition

Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ be a weighted tree. A function $u : \mathcal{E} \to \mathbb{R}$ will be in $\mathrm{L}^2_\mu(\mathcal{T})$ if and only if $\mu |u|^2 \in L^1(\mathcal{E})$:

$$\|u\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} = \int_{\mathcal{T}} \mu(\mathbf{x})|u(\mathbf{x})|^2 d\mathbf{x} := \sum_{(\ell,j)\in\mathbb{E}_p} \int_{e_{\ell,j}} \mu(\mathbf{x})|u(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

Sobolev spaces

Definition

Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ *be a weighted tree.*

$$\mathcal{H}^1_\mu(\mathcal{T}) = \left\{ u \in \mathrm{L}^2_{\mu,loc}(\mathcal{E}) \cap C(\mathcal{E}) \ / \ u' \in \mathrm{L}^2_\mu(\mathcal{T}) \right\}.$$

This space is an Hilbert space with associated norm

$$\|u\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = \left|u(v_{0,0})\right|^2 + |u|^2_{\mathcal{H}^1_\mu(\mathcal{T})}, \quad |u|_{\mathcal{H}^1_\mu(\mathcal{T})} = \left\|u'\right\|_{\mathrm{L}^2_\mu(\mathcal{T})}. \tag{1}$$

Rk. $1 \in \mathcal{H}^1_\mu(\mathcal{T})$.

Definition

Let $\mathcal{T} = (\mathcal{E}, \mathcal{V}, \mu)$ *be a weighted tree.*

- $\mathcal{H}^1_{\mu,c}(\mathcal{T})$ = *subset of functions* $u \in \mathcal{H}^1_\mu(\mathcal{T})$ *whose support is compact,*
- $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ *the closure of* $\mathcal{H}^1_{\mu,c}(\mathcal{T})$ *in* $\mathcal{H}^1_\mu(\mathcal{T})$ *for the norm (1).*

The question

On which condition over the triplet $(\mathcal{E}, \mathcal{V}, \mu)$, $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$?

For some particular trees and weights in the finite difference version, see again

B. Maury, D. Salort, and C. Vannier.
Trace theorems for trees, application to the human lungs.
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A first implicit NSC

Theorem (Thm 1)

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad 1 \in \mathcal{H}^1_{\mu,0}(\mathcal{T}) \tag{2}$$

Proof.

$\Rightarrow$: trivial.
$\Leftarrow$: By assumption, $\exists v_n \in \mathcal{H}^1_{\mu,c}(\mathcal{T})$ s.t. $v_n \to 1$ in $\mathcal{H}^1_\mu(\mathcal{T})$.
For any $u \in \mathcal{H}^1_\mu(\mathcal{T})$, we build up a sequence $u_n \in \mathcal{H}^1_{\mu,c}(\mathcal{T})$ (using v_n and u) s.t. $u_n \to u$ in $\mathcal{H}^1_\mu(\mathcal{T})$. □

Auxiliary Dirichlet problems

$(\mathcal{P}_{\mathrm{D}})$ Find $u_{\mathrm{D}} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ such that $u_{\mathrm{D}}(v_{0,0}) = 1$ and

$$\int_{\mathcal{T}} \mu(\mathbf{x}) u'_{\mathrm{D}}(\mathbf{x}) \overline{\phi'(\mathbf{x})} d\mathbf{x} = 0, \quad \forall \phi \in \mathcal{H}^1_{\mu,0}(\mathcal{T}),\ \phi(v_{0,0}) = 0.$$

For all $n \in \mathbb{N}$, introduce the following spaces:

$$\mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T}) = \left\{ u \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T}) \text{ such that } \operatorname{supp} u \subset \mathcal{T}^n \right\},$$

$$\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T}) = \left\{ u \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T}) \text{ such that } u(v_{0,0}) = 0 \text{ and } \operatorname{supp} u \subset \mathcal{T}^n \right\}.$$

$(\mathcal{P}_{\mathrm{D,n}})$ Find $u^n \in \mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T})$ such that $u^n(v_{0,0}) = 1$ and

$$\int_{\mathcal{T}} \mu(\mathbf{x}) (u^n)'(\mathbf{x}) \overline{\phi'(\mathbf{x})} d\mathbf{x} = 0, \quad \forall \phi \in \mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T}).$$

A reformulation of the first NSC

Proposition (Prop 2)

We have the following equivalence

$$1 \textit{ is solution of } (\mathcal{P}_{\mathrm{D}}) \iff 1 \in \mathcal{H}^1_{\mu,0}(\mathcal{T}) \tag{3}$$

Some properties of u^n

$$u^n \to u_{\mathrm{D}} \text{ in } \mathcal{H}^1_\mu(\mathcal{T}) \text{ as } n \to \infty.$$

$$0 \leq u^n \leq 1 \text{ (maximum principle).}$$

$$(u^n_{\ell,j})' \leq 0, \forall \ell, j.$$

$$|u^n|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = -\mu_{0,0}(u^n)'(v_{0,0}). \tag{4}$$

Relation with the Liouville property

Definition

A weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ is called a Liouville network if and only if every bounded harmonic function on $\mathcal{T}$ is constant.

For $\mu = 1$, see

J. von Below and J. A. Lubary.
Harmonic functions on locally finite networks.
Results Math., 45(1-2):1–20, 2004.

Proposition

We have the following equivalence

$$u_D = 1 \iff \mathcal{T} \text{ is a Liouville network.} \qquad (5)$$

Proof.

The implication $\Rightarrow$ is direct, since u_D is harmonic and bounded (since $0 \leq u_D \leq 1$).
$\Leftarrow$: Fix a bounded harmonic function h on $\mathcal{T}$. As the assumption is that $u_D = 1$, by Prop. 2, this is equivalent to $1 \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$. Hence let us fix a sequence of functions $(v_n)_{n\in\mathbb{N}} \in \mathcal{H}^1_{\mu,c}(\mathcal{T})$ s. t.

$$\|1 - v_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0 \quad \text{as} \quad n \to \infty.$$

Then the result is based on (consequence of Green's formula)

$$\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 v_n^2(\mathbf{x}) d\mathbf{x} = -2 \int_{\mathcal{T}} \mu(\mathbf{x}) h'(\mathbf{x}) v_n(\mathbf{x}) v_n'(\mathbf{x}) h(\mathbf{x}) d\mathbf{x} + \mu_{0,0} h'(v_{0,0}) v_n^2(v_{0,0}) h(v_{0,0}).$$

□

A third implicit NSC

Theorem (Thm 3)

One has

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \lim_{n\to\infty} |u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} = 0.$$

Proof.

$\Leftarrow$: Since $|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0$ and $u^n(v_{0,0}) = 1$:

$$\lim_{n\to\infty} \|u^n - 1\|_{\mathcal{H}^1_\mu(\mathcal{T})} = 0.$$

As $u^n \in \mathcal{H}^1_{\mu,c}(\mathcal{T})$, one gets that $1 \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$, and $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$ by using Thm 1.

$\Rightarrow$: We use a contradiction argument and Prop 2. □

An electrical problem

On each edge $e_{\ell,j}$, we introduce the resistance $R_{\ell,j} = \int_{e_{\ell,j}} \frac{d\mathbf{x}}{\mu(\mathbf{x})}$, and the new unknowns

$$U^n_{\ell,j} = u^n(v_{\ell+1,j}) - u^n(v_{\ell,\lfloor p^{-1}j\rfloor}), \; I^n_{\ell,j} = \mu_{\ell,j}(u^n)'(v_{\ell+1,j}).$$

This new set of unknowns $(U^n_{\ell,j}, I^n_{\ell,j})$ allows us to re-write $(\mathcal{P}_{\mathrm{D,n}})$ in the following equivalent form:

- on each edge $e_{\ell,j}$, $I^n_{\ell,j}$ is constant and $U^n_{\ell,j} = R_{\ell,j} I_{\ell,j}$,
- for any $j \in \{0, \ldots, p^n - 1\}$, we have $\sum_{\ell=0}^{n} U^n_{\ell,\lfloor p^{\ell-n} j\rfloor} = -1$,
- for any $0 \leq \ell \leq n-1$, we have $I^n_{\ell,j} = \sum_{k=0}^{p-1} I^n_{\ell+1,pj+k}$ (Kirchoff law).

We have actually rewritten problem $(\mathcal{P}_{\mathrm{D,n}})$ as a general electrical problem.

If R^n is the equivalent resistance of the finite tree $\mathcal{T}^n$, Ohm law $\Rightarrow$

$$-1 = R^n I^n_{0,0}.$$

The definition of $I^n_{0,0}$ and relation (4) $\Rightarrow$

$$|u^n|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = (R^n)^{-1}. \tag{6}$$

An explicit NSC

Theorem

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \iff \lim_{n\to\infty} R^n = +\infty.$$

Proof.

We use Thm 3 and the relation (6). □

Proposition (Equivalent resistance)

$$R^n = R_{0,0} + \left(\sum_{j_1=0}^{p-1}\left(R_{1,j_1} + \left(\sum_{j_2=0}^{p-1}\left(R_{2,pj_1+j_2} + \dots \left(\sum_{j_n=0}^{p-1} R^{-1}_{n,\sum_{k=1}^{n} p^{n-k}j_k}\right)^{-1}\right)^{-1}\right)\right)^{-1}$$

Proof.

Induction on n, all resistances of the last generation are in parallel. □

An example

Theorem

Let us assume that the ratios $\mu_{\ell,j}/L_{\ell,j}$ are constant and equal on the same generation, i. e., for all $\ell \in \mathbb{N}$ there exists $\nu_\ell > 0$ s. t.

$$\mu_{\ell,j}/L_{\ell,j} = \nu_\ell, \forall j = 0, \cdots, p^\ell - 1.$$

Then

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \iff \sum_{\ell=0}^\infty p^{-\ell}\nu_\ell^{-1} \text{ diverges.}$$

Rk In the case $p = 2$, we obtain the continuous version of Theorem 2.12 of

B. Maury, D. Salort, and C. Vannier.
Trace theorems for trees, application to the human lungs.
Network and Heteregeneous Media, 4(3):469 – 500, 2009.

A recursive partition

Definition

We call p-recursive partition of the unit interval a sequence of real numbers $\gamma^n_{p,j} \in [0,1]$ defined for $n \geq 1$ and $0 \leq j \leq p^{n-1}$ such that

- $\gamma^n_{p,0} = 0, \gamma^n_{p,p^{n-1}} = 1$ *for any* $n \geq 1$,
- $\gamma^n_{p,j} \leq \gamma^n_{p,j+1}$, *for all* $0 \leq j < p^{n-1}$,
- $\gamma^{n+1}_{p,pj} = \gamma^n_{p,j}$ *for any* $n \geq 1$ *and for any* $0 \leq j \leq p^{n-1}$.

Rk • the intervals $]\gamma^n_{p,j}, \gamma^n_{p,j+1}[$, $0 \leq j < p^{n-1}$: subdivision of $]0,1[$,
• the intervals $]\gamma^{n+1}_{p,pj+k}, \gamma^{n+1}_{p,pj+k+1}[$, $0 \leq k < p$: subdivision of $]\gamma^n_{p,j}, \gamma^n_{p,j+1}[$.

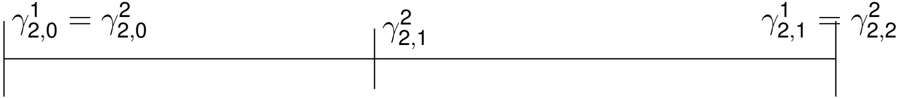

Figure: Example of a 2-recursive partition

Trace operators

Definition

Given $\gamma = (\gamma^n_{p,j})_{n\geq 1, 0\leq j\leq p^{n-1}}$ *a p-recursive partition of the unit interval, we can define the following trace operators:*

- *the n-th trace operator* T^n_γ*: for* $u \in \mathcal{H}^1_\mu(\mathcal{T})$*, we define* $T^n_\gamma(u) \in \mathrm{L}^2(]0,1[)$ *as*

$$T^n_\gamma(u)(x) = u(v_{n,j}), \quad x \in \left]\gamma^n_{p,j}, \gamma^n_{p,j+1}\right[, \tag{7}$$

- *the trace operator* $T^\infty_\gamma(u)$ *defined as the limit of* $T^n_\gamma(u)$ *in* $\mathrm{L}^2(]0,1[)$*, as n goes to infinity (if it exists).*

Rk For any choose of γ, $T^\infty_\gamma(1) = 1$.

A negative result

Proposition

If $1 \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$*, then there exists no p-recursive partition* γ *of the unit interval such that* T^∞_γ *is continuous from* $\mathcal{H}^1_\mu(\mathcal{T})$ *to* $\mathrm{L}^2(]0,1[)$*.*

Proof.

Let γ be any p-recursive partition of the unit interval. By hypothesis, there exists a sequence $(u_n) \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ such that

$$\|1 - u_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0, \quad \text{as } n \to \infty.$$

As for a given n, $T^\infty_\gamma(1 - u_n) = 1$,

$$\left\| T^\infty_\gamma(1 - u_n)\right\|_{\mathrm{L}^2(]0,1[)} = 1 \not\to 0, \quad \text{as } n \to \infty.$$

□

A trace theorem

Theorem

Given a weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$*, and assume that*

$$R = \lim_{n\to\infty} R^n < \infty.$$

Then $\exists$ *a p-recursive partition* γ *of the unit interval such that* T_γ^∞ *is continuous from* $\mathcal{H}^1_\mu(\mathcal{T})$ *to* $\mathrm{L}^2(]0,1[)$*.*

Proof.

Based on the identity

$\int_{\mathcal{T}^n} \mu(\mathbf{x}) u'_{\mathrm{D}}(\mathbf{x})(u^2)'(\mathbf{x}) d\mathbf{x} =$
$\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 - \mu_{0,0} u'_{\mathrm{D}}(v_{0,0}) |u(v_{0,0})|^2.$
consequence of Green's formula. Then we take

$$\ell^n_{p,j} = -R\mu_{n-1,j} u'_{\mathrm{D},n-1,j}, \;\; \gamma^n_{p,0} = 0, \gamma^n_{p,j} = \sum_{k<j} \ell^n_{p,k}, \forall j \geq 1.$$


Thank you