Dynamics of a Mirror and the Electromagnetic Field

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• L O. Castaños and R. Weder, *Classical dynamics of a thin moving mirror interacting with a laser*, Phys. Rev. A **89** 06387 (2014) 21 pp.

• L. O. Castaños and R. Weder, *Equations of a Moving Mirror and the Electromagnetic Field*, Phys. Scrip. **90** 068011 (2015) 16pp.

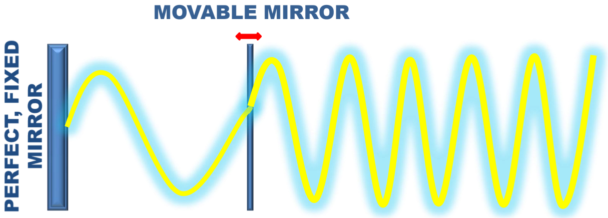
• L. O. Castaños, R. Weder, How the electromagnetic field evolves in the presence of a mobile membrane, arXiv: 1606.07113 [physics.optics] (2016), 31 pages. Physical Review A **94** 033846 (2016).

SYSTEM

- **One-dimensional cavity** composed of
  - Perfect, fixed mirror at x=0.
  - Thin, movable mirror at x=q(t) .

    The mirror is made of a material that is linear, isotropic, non-magnetizable, and conducting.

- **Movable mirror subject only to radiation pressure.**



The Mirror

The mirror is made of a material that is linear, isotropic, non-magnetizable, conducting and neutral, when it is at rest.

We denote by $\chi(x)$ and $\sigma(x)$, respectively, the electric susceptibility and the conductivity of the mirror at rest.
They are (piecewise) continuously differentiable functions that are different from zero only for $|x| \leq \delta_0/2$. The tickness of the mirror at rest is δ_0.

At time t the mirror occupies the region in the laboratory system,

$$R(t) = \left\{ \mathbf{r} \in \mathbb{R}^3 : \ |x - q(t)| \leq \frac{\delta(t)}{2} \right\}.$$

Here $q(t)$ and $\delta(t)$ are the midpoint and the thickness of the mirror along the x-axis, respectively.

Notice that $\delta(t)$ depends on time due to the Lorentz contraction because it can be in motion.

We refer to $q(t)$ and $\delta(t)$ as the midpoint and thickness of the mirror in the laboratory system, respectively. The midpoint has velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$ with derivatives with respect to t denoted by a dot.

The Equations for the Field

$$\begin{aligned}
\nabla \times \mathbf{B}(x,t) &= \frac{4\pi}{c}\left[\mathbf{J}_f(x,t) + \mathbf{J}_b(x,t)\right] + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}(x,t) , \\
\nabla \times \mathbf{E}(x,t) &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}(x,t) , \\
\nabla \cdot \mathbf{E}(x,t) &= 4\pi\left[\rho_f(x,t) + \rho_b(x,t)\right] , \\
\nabla \cdot \mathbf{B}(x,t) &= 0 .
\end{aligned}$$

We need to compute $\mathbf{J}_f(x,t), \mathbf{J}_b(x,t), \rho_f(x,t), \rho_b(x,t)$.

In the reference system where the mirror is instantaneously at rest we have that

$$\mathbf{P}''(x'', t'') = \chi(x'')\mathbf{E}''(x'', t''), \qquad \mathbf{M}''(x'', t'') = \mathbf{0},$$

$$\rho_f''(x'', t'') = 0, \qquad \mathbf{J}_f''(x'', t'') = \sigma(x'')\mathbf{E}''(x'', t''),$$

where $\chi(x)$ is the electric susceptibilty and $\sigma(x)$ the conductivity. We wish to compute $\mathbf{J}_f(x, t), \mathbf{J}_b(x, t), \rho_f(x, t), \rho_b(x, t)$ in the laboratory system by a relativistic treatment by a Lorentz transformation, from a instantaneous inertial frame where the mirror is approximately at rest, to the laboratory frame

Why a relativistic treatment if the velocity of the mirror is usually small ? Well:

1.The electromagnetic quantities have a simple and well defined transformation properties under Lorentz transformations.

2.The use of Lorentz transformations allows us to better understand the physics of the problem and it gives us corrections terms in the case of small velocities and accelerations.

3. Our relativistic treatment allows us to obtain consistent approximations for the equations of both the field and the slab at any given order of the velocity and the acceleration.

4. Our treatment allows the slab to accelerate slowly up to relativistic velocities, and its motion is not restricted to oscillations, it can go far from its equilibrium position.

The Laboratory System **LS**

1. The coordinate axes are defined by unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ and these form a right-handed system.
2.The coordinates of an arbitrary event are denoted by (x, y, z, ict). Also, $\mathbf{r} \equiv (x, y, z)$.
3. At time t the mirror occupies the region

$$R(t) \; = \; \left\{ \mathbf{r} \in \mathbb{R}^3 : \; |x - q(t)| \leq \frac{\delta(t)}{2} \right\}.$$

Here $q(t)$ and $\delta(t)$ are the midpoint and the thickness of the mirror along the x-axis in **LS**, respectively. The midpoint has velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$.
4. There is empty space outside of $R(t)$.
5. The electric and magnetic fields are: $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$. The polarization, magnetization, free current density, and free charge density are: $\mathbf{P}(x, t)$, $\mathbf{M}(x, t)$, $\mathbf{J}_f(x, t)$, and $\rho_f(x, , t)$, respectively.

The system $\mathbf{LS}_0$

Let $t_0 \in \mathbb{R}$ be fixed. Consider an inertial reference frame $\mathbf{LS}_0$ obtained from **LS** by a time and space translation where t_0 is the new origin of time and $(q(t_0), 0, 0)$ is the new origin of space. In $\mathbf{LS}_0$ one has the following properties:
1. The coordinate axes are defined by unit vectors $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, and $\hat{\mathbf{z}}'$ parallel respectively to $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.
2. The coordinates of an arbitrary event are denoted by (x', y', z', ict') and they are related to the corresponding coordinates (x, y, z, ict) in **LS** by

$$\begin{aligned} x' &= x - q(t_0) \,, \quad y' = y \,, \\ t' &= t - t_0 \,, \qquad z' = z \,. \end{aligned}$$

Also, $\mathbf{r}' \equiv (x', y', z')$.

3. The electric and magnetic fields are denoted by $\mathbf{E}'(x', t')$ and $\mathbf{B}'(x', t')$, while the polarization, magnetization, free current density, and free charge density are denoted by $\mathbf{P}'(x', t')$, $\mathbf{M}'(x', t')$, $\mathbf{J}'_f(x', t')$, and $\rho'_f(x', t')$, respectively.

Notice that a quantity $f'(x', t')$ in $\mathbf{LS}_0$ is connected to the corresponding quantity $f(x, t)$ in $\mathbf{LS}$ by

$$f'(x', t') = f\left[x' + q(t_0), t' + t_0\right].$$

Observe that at time t' the mirror occupies the region

$$R'(t') \;=\; \left\{ \mathbf{r}' \in \mathbb{R}^3 : \; \left| x' - q'(t') \right| \le \frac{\delta'(t')}{2} \right\} ,$$

with

$$q'(t') = q(t' + t_0) - q(t_0) \; , \quad \delta'(t') = \delta(t' + t_0) \; ,$$

its midpoint and its thickness along the x'-axis. Also, there is empty space outside of $R'(t')$.

In particular, at time $t' = 0$ the mirror occupies the region

$$R'(0) \;=\; \left\{ \mathbf{r}' \in \mathbb{R}^3 : \; -\frac{\delta(t_0)}{2} \le x' \le \frac{\delta(t_0)}{2} \; \right\} ,$$

and its mid-point satisfies

$$q'(0) = 0 \; , \;\; \frac{dq'}{dt'}(0) = \dot{q}(t_0) \; , \;\; \frac{d^2 q'}{dt'^2}(0) = \ddot{q}(t_0) \; . \qquad (1)$$

The Mirror System $\mathbf{MS}_0$

Define

$$v_0 = \dot{q}(t_0)\ , \quad \beta_0 = \frac{v_0}{c}\ , \quad \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}\ .$$

Notice that v_0 and, consequently, β_0 can be positive or negative.
Consider an inertial reference frame $\mathbf{MS}_0$ (for *Mirror System*) where

1. The coordinate axes are defined by unit vectors $\hat{\mathbf{x}}''$, $\hat{\mathbf{y}}''$, and $\hat{\mathbf{z}}''$ parallel respectively to $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, and $\hat{\mathbf{z}}'$.
2. $\mathbf{MS}_0$ moves with velocity $v_0\mathbf{x}'$ with respect to $\mathbf{LS}_0$.
3. The coordinates of an arbitrary event are denoted by (x'', y'', z'', ict'') and they are related to the corresponding coordinates (x', y', z', ict') in $\mathbf{LS}_0$ by a Lorentz transformation

$$\begin{aligned} ct'' &= \gamma_0(ct' - \beta_0 x')\ , \qquad & y'' &= y'\ , \\ x'' &= \gamma_0(x' - \beta_0 ct')\ , \qquad & z'' &= z'\ . \end{aligned}$$

Here the space-time origin ($\mathbf{r}' = \mathbf{0}, ict' = 0$) of $\mathbf{LS}_0$ coincides with the space-time origin ($\mathbf{r}'' = \mathbf{0}, ict'' = 0$) of $\mathbf{MS}_0$. Also, $\mathbf{r}'' \equiv (x'', y'', z'')$.

4. The electric and magnetic fields are denoted by $\mathbf{E}''(x'', t'')$ and $\mathbf{B}''(x'', t'')$, while the polarization, magnetization, free current density, and free charge density are denoted by $\mathbf{P}''(x'', t'')$, $\mathbf{M}''(x'', t'')$, $\mathbf{J}''_f(x'', t'')$, and $\rho''_f(x'', t'')$, respectively. Using the Lorentz transformation one can relate the coordinates of the midpoint $q'(t')$ in $\mathbf{LS}_0$ with those of the midpoint $q''(t'')$ of the mirror along the x''-axis in $\mathbf{MS}_0$.

Consider the event whose coordinates in $\mathbf{LS}_0$ are given by

$$b' = \left(q'(t'), y', z', ict'\right) .$$

b' is an event associated with a midpoint of the mirror. This event has coordinates in $\mathbf{MS}_0$ given by

$$b'' = \left(q''(t''), y', z', ict''\right) ,$$

with

$$\begin{aligned} q''(t'') &= \gamma_0 \left[q'(t') - \beta_0 ct'\right] , \\ ct'' &= \gamma_0 \left[ct' - \beta_0 q'(t')\right] . \end{aligned}$$

$$\begin{aligned} t'' &= 0 , \quad \frac{dq''}{dt''}(0) = 0 , \\ q''(0) &= 0 , \quad \frac{d^2 q''}{dt''^2}(0) = \gamma_0^3 \ddot{q}(t_0) . \end{aligned}$$

Therefore, the midpoint $q''(t'')$ of the mirror is at rest at the coordinate origin in $\mathbf{MS}_0$ at time $t'' = 0$, although it can have a non-zero acceleration. Note that, in general, the other points of the mirror are not necessarily at rest in $\mathbf{MS}_0$ at time $t'' = 0$.

It follows that the mirror occupies the following region in $\mathbf{MS}_0$ at time $t'' = 0$:

$$R''(0) \;=\; \left\{ \mathbf{r}'' \in \mathbb{R}^3 : \; |x''| \leq \frac{\delta_0}{2} \right\}.$$

By the Lorentz transformation the following relation holds :

$$x' \in \left[-\frac{\delta(t_0)}{2}, \frac{\delta(t_0)}{2}\right], \quad t' = 0$$

$$\Rightarrow \begin{cases} x'' \in \left[-\gamma_0 \frac{\delta(t_0)}{2}, \gamma_0 \frac{\delta(t_0)}{2}\right], \\ t'' \in \left[-\gamma_0 |\beta_0| \frac{\delta(t_0)}{2c}, \gamma_0 |\beta_0| \frac{\delta(t_0)}{2c}\right] \end{cases} .$$

In other words, points x' in the mirror at time $t' = 0$ in **LS_0** correspond in **MS_0** to points x'' in the mirror at some time t'' in the interval on the right-hand side of the equation above. If we assume that the mirror is at rest in **MS_0** for all t'' in the aforementioned interval, then we will be able to use the *rest properties* of the mirror for (x'', t'') corresponding to (x', t') such that x' is inside the mirror and $t' = 0$.

We assume that the mirror is (approximately) at rest in **MS_0** during the time interval

$$\left[-t_1'', t_1''\right], \quad t_1'' = \gamma_0 |\beta_0| \frac{\delta(t_0)}{2c}, \quad \delta(t_0) = \frac{\delta_0}{\gamma_0} .$$

Then, we can use the transformation rules of the electric and magnetic fields, of the polarization and the magnetization and of the free charge and the free current to compute them in the laboratory system from the corresponding values in the mirror system.

$$\begin{pmatrix} \mathbf{E}'(x',t') \\ \mathbf{B}'(x',t') \end{pmatrix} = \mathbb{M}_0 \begin{pmatrix} \mathbf{E}''(x'',t'') \\ \mathbf{B}''(x'',t'') \end{pmatrix} ,$$

where

$$\mathbb{M}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_0 & 0 & 0 & 0 & \gamma_0\beta_0 \\ 0 & 0 & \gamma_0 & 0 & -\gamma_0\beta_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma_0\beta_0 & 0 & \gamma_0 & 0 \\ 0 & \gamma_0\beta_0 & 0 & 0 & 0 & \gamma_0 \end{pmatrix} .$$

In the mirror system, during the time interval where the mirror is (approximately) at rest

$$\begin{aligned} \mathbf{P}''(x'', t'') &= \chi(x'')\mathbf{E}''(x'', t'') , \\ \mathbf{M}''(x'', t'') &= \mathbf{0} \qquad \text{for all } x'' \text{ and } t'' \in [-t_1'', t_1'']. \end{aligned}$$

Moreover,

$$\begin{pmatrix} \mathbf{P}'(x', t') \\ \mathbf{M}'(x', t') \end{pmatrix} = \mathbb{M}_0^{-1} \begin{pmatrix} \mathbf{P}''(x'', t'') \\ \mathbf{M}''(x'', t'') \end{pmatrix} .$$

Since the mirror is electrically neutral and satisfies Ohm's law when it is at rest and the mirror is (approximately) at rest during the time interval $[-t_1'', t_1'']$, one has

$$\rho_f''(x'', t'') = 0 \ , \quad \mathbf{J}_f''(x'', t'') = \sigma(x'')\mathbf{E}''(x'', t'') \ ,$$

for all x'' and $t'' \in [-t_1'', t_1'']$. The current four-vector in $\mathbf{MS}_0$ is given by

$$\begin{aligned} \mathbf{s}''(x'', t'') &= \left(\mathbf{J}_f''(x'', t''), \ ic\rho_f''(x'', t'')\right)^T \ , \\ &= \left(\sigma(x'')\mathbf{E}''(x'', t''), \ 0\right)^T \ , \end{aligned}$$

for all x'' and $t'' \in [-t_1'', t_1'']$.

We have that,

$$\begin{aligned} \mathbf{s}'(x',t') &= \left(\mathbf{J}'_f(x',t'),\ ic\rho'_f(x',t')\right)^T , \\ &= \mathbb{M}_1 \mathbf{s}''(x'',t'') , \end{aligned}$$

with

$$\mathbb{M}_1 = \begin{pmatrix} \gamma_0 & 0 & 0 & -i\gamma_0\beta_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\gamma_0\beta_0 & 0 & 0 & \gamma_0 \end{pmatrix} .$$

In this way, using the Lorentz transformations we prove that,

$$\begin{aligned}\mathbf{P}(x,t) &= \gamma(t)^2\chi_{\text{LS}}(x,t)\left[\mathbf{E}(x,t)+\beta(t)\mathbf{x}\times\mathbf{B}(x,t)\right. \\ &\qquad \left. -\beta(t)^2E_1(x,t)\mathbf{x}\right], \\ \mathbf{M}(x,t) &= -\beta(t)\mathbf{x}\times\mathbf{P}(x,t),\end{aligned}$$

where

$$\beta(t) \equiv \frac{\dot{q}(t)}{c}, \qquad \gamma(t) \equiv \frac{1}{\sqrt{1-\beta(t)^2}},$$

and

$$\chi_{\text{LS}}(x,t) = \chi\left\{\gamma(t)\left[x-q(t)\right]\right\}.$$

Notice that although the mirror is linear, isotropic, and non-magnetizable when it is at rest, to an observer in **LS** it appears to have a magnetization and the polarization depends not only on the electric field, but also on the velocity of the mirror and on the magnetic field.

Furthermore, the free charges and the free current in **LS** are given by:

$$\begin{aligned} \mathbf{J}_f(x,t) &= \gamma(t)\sigma_{\mathrm{LS}}(x,t)\left[\mathbf{E}(x,t) + \beta(t)\mathbf{x} \times \mathbf{B}(x,t)\right] , \\ \rho_f(x,t) &= \frac{\gamma(t)}{c}\beta(t)\sigma_{\mathrm{LS}}(x,t)E_1(x,t) , \end{aligned}$$

with

$$\sigma_{\mathrm{LS}}(x,t) = \sigma\left\{\gamma(t)\left[x - q(t)\right]\right\} .$$

Notice that, even if the mirror is electrically neutral when it is at rest, it appears to be charged to an observer in **LS** if $E_1(x,t) \neq 0$. Also, observe that the mirror does not satisfy Ohm's law when it is in motion.

Recalling that the bound charge and the bound current are given by:

$$\rho_b(x,t) \equiv -\nabla \cdot \mathbf{P}(x,t) ,$$

$$\mathbf{J}_b(x,t) \equiv \frac{\partial \mathbf{P}}{\partial t}(x,t) + c\nabla \times \mathbf{M}(x,t) ,$$

we have all the quantities that we need in the Maxwell equations.

$$\begin{aligned}
\nabla \times \mathbf{B}(x,t) &= \frac{4\pi}{c}\left[\mathbf{J}_f(x,t) + \mathbf{J}_b(x,t)\right] + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}(x,t) , \\
\nabla \times \mathbf{E}(x,t) &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}(x,t) , \\
\nabla \cdot \mathbf{E}(x,t) &= 4\pi\left[\rho_f(x,t) + \rho_b(x,t)\right] , \\
\nabla \cdot \mathbf{B}(x,t) &= 0 .
\end{aligned}$$

The Case of Transverse Fields

We assume that the electromagnetic field can be deduced from the potentials $\mathbf{A}(x,t) = A_0(x,t)\,\mathbf{z}$, $V(x,t) = 0$. Then,

$$\mathbf{B}(x,t) = \nabla \times \mathbf{A}(x,t) = -\frac{\partial A_0}{\partial x}(x,t)\mathbf{y},$$

$$\mathbf{E}(x,t) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}(x,t) = -\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\mathbf{z}$$

$$\alpha_0(x,t)\frac{\partial^2 A_0}{\partial x^2}(x,t) = \frac{\alpha_1(x,t)}{c^2}\frac{\partial^2 A_0}{\partial t^2}(x,t) + \frac{\alpha_2(x,t)}{c}\frac{\partial^2 A_0}{\partial x \partial t}(x,t) + \frac{\alpha_3(x,t)}{c}\frac{\partial A_0}{\partial x}(x,t) + \frac{\alpha_4(x,t)}{c^2}\frac{\partial A_0}{\partial t}(x,t),$$

where

$$\alpha_0(x,t) = 1 - 4\pi\gamma(t)^2\beta(t)^2\chi_{\text{LS}}(x,t), \quad \alpha_1(x,t) = 1 + 4\pi\gamma(t)^2\chi_{\text{LS}}(x,t)$$

$$\alpha_2(x,t) = 8\pi\gamma(t)^2\beta(t)\chi_{\text{LS}}(x,t)$$

$$\alpha_3(x,t) = 4\pi \frac{d\beta}{dt}(t)\left[\gamma(t)^2\chi_{\rm LS}(x,t) + \beta(t) f_0(x,t)\right] + 4\pi\gamma(t)\sigma_{\rm LS}(x,t)\beta(t)$$

$$\alpha_4(x,t) = 4\pi f_0(x,t)\frac{d\beta}{dt}(t) + 4\pi\gamma(t)\sigma_{\rm LS}(x,t)$$

with

$$f_0(x,t) = \gamma(t)^4\beta(t)\left\{2\chi(x'') + x''\frac{d\chi}{dx''}(x'')\right\}, \qquad x'' = \gamma(t)[x - q(t)].$$

The Force on the Mirror

For our transverse fields the free and the bound charges are zero, $\rho_f = 0, \rho_b = 0$.
Consider the volume

$$V(t) = \left[q(t) - \frac{\delta(t)}{2},\ q(t) + \frac{\delta(t)}{2} \right] \times [y_0, y_1] \times [z_0, z_1] ,$$

with $y_0 < y_1$ and $z_0 < z_1$.
The force on the mirror in $V(t)$ at time t is given by

$$\mathbf{F}(t) = \int_{V(t)} d^3 r \frac{1}{c} \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) .$$

Using the expression for the force and a straightforward calculation we prove that the pressure, $f(t)$, exerted by the field along the x axis is,

$$f(t) = -\frac{1}{8\pi}\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\Bigg|_{x=q(t)-\delta(t)/2}^{x=q(t)+\delta(t)/2}$$
$$+\frac{1}{4\pi c^2}\int_{q(t)-\delta(t)/2}^{q(t)+\delta(t)/2} dx\,\frac{\partial^2 A_0}{\partial t^2}(x,t)\frac{\partial A_0}{\partial x}(x,t).$$

Let $p(t)$ be the mechanical momentum of the mirror (along the x-axis) per unit area perpendicular to the x-axis,

$$p(t) = M_0\gamma(t)\dot{q}(t),$$

where M_0 is the mass per unit area of the mirror at rest.

The Equation for the mirror

Then, the equation for the mirror is given by

$$\frac{d}{dt}p(t) = f(t).$$

The equation for the mirror combined with the equation for the field constitutes the self-consistent set of equations governing the dynamics of the mirror-field system. The basic assumption for these equations to hold is that the acceleration of the mirror is small

First Order Equations

The following first order equations are correct to first order in the velocity and the acceleration of the mirror.

$$
\begin{aligned}
\frac{\partial^2 A_0}{\partial x^2}(x,t) \quad = \quad & \frac{1+4\pi\chi\left[x-q(t)\right]}{c^2}\frac{\partial^2 A_0}{\partial t^2}(x,t) \\
& +4\pi\frac{\sigma\left[x-q(t)\right]}{c^2}\frac{\partial A_0}{\partial t}(x,t) \\
& +8\pi\frac{\chi\left[x-q(t)\right]}{c^2}\dot{q}(t)\frac{\partial^2 A_0}{\partial x\partial t}(x,t) \\
& +4\pi\frac{\sigma\left[x-q(t)\right]}{c^2}\dot{q}(t)\frac{\partial A_0}{\partial x}(x,t) \\
& +4\pi\frac{\chi\left[x-q(t)\right]}{c^2}\ddot{q}(t)\frac{\partial A_0}{\partial x}(x,t)\ .
\end{aligned}
$$

$$M\left[q(t),t\right]\ddot{q}(t) = F_0\left[q(t),t\right] - F_1\left[q(t),t\right]\frac{\dot{q}(t)}{c} \ ,$$

with

$$\begin{aligned} F_0 &= -\frac{1}{2}\int_{q(t)-\frac{\delta_0}{2}}^{q(t)+\frac{\delta_0}{2}} dx\, \frac{\chi[x-q(t)]}{1+4\pi\chi[x-q(t)]}\times \\ &\quad \times\frac{\partial}{\partial x}\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2 \ , \\ &\quad -\frac{1}{c^2}\int_{q(t)-\frac{\delta_0}{2}}^{q(t)+\frac{\delta_0}{2}} dx\, \frac{\sigma[x-q(t)]}{1+4\pi\chi[x-q(t)]}\times \\ &\quad \times\frac{\partial A_0}{\partial x}(x,t)\frac{\partial A_0}{\partial t}(x,t) \ , \\ F_1 &= \frac{1}{c}\int_{q(t)-\frac{\delta_0}{2}}^{q(t)+\frac{\delta_0}{2}} dx\left\{\frac{\sigma[x-q(t)]}{1+4\pi\chi[x-q(t)]}\right. \\ &\quad \left. +\frac{\chi[x-q(t)]}{1+4\pi\chi[x-q(t)]}\frac{\partial}{\partial t}\right\}\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2 \end{aligned}$$

$$M = M_0 + \frac{1}{c^2} \int_{q(t)-\frac{\delta_0}{2}}^{q(t)+\frac{\delta_0}{2}} dx \, \frac{\chi[x - q(t)]}{1 + 4\pi\chi[x - q(t)]} \times \left[\frac{\partial A_0}{\partial x}(x,t) \right]^2 .$$

The mirror has a position and time dependent mass and it is subject to a friction force proportional to the velocity.

This system is currently being analyzed with a multiple scale method

The first order equation for the field was derived in Cheung H K and Law C K, *Phys. Rev. A* **84**, (2011) 023812 (see also references therein), from an approximate Lagrangian in the case where $\chi(x) = \chi_0$ for, $-\delta_0/2 \leq x \leq \delta_0/2$ and zero elsewhere. Note however that this approximate Lagrangian does not give the first order equation for the mobile mirror, that appears to be new.

Equations at leading order

The simplest case for the dynamics of the mirror+field system is to consider a non-conducting material (that is, $\sigma = 0$) with $\chi(x) = \chi_0$ for, $-\delta_0/2 \leq x \leq \delta_0/2$ and zero elsewhere, and to make an approximation to order zero in both the velocity and the acceleration of the mirror, in the equation for the field and in the force affecting the mirror. In this case we obtain that

$$M_0 \ddot{q}(t) = -\frac{1}{2}\left(\frac{\chi_0}{1+4\pi\chi_0}\right)\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\Bigg|_{x=q(t)-\frac{\delta_0}{2}}^{x=q(t)+\frac{\delta_0}{2}},$$

$$\frac{\partial^2 A_0}{\partial x^2}(x,t) = \frac{1+4\pi\chi[x-q(t)]}{c^2}\frac{\partial^2 A_0}{\partial t^2}(x,t).$$

Taking the limits limits: $\delta_0 \downarrow 0$, $\rho_{M0} \to +\infty$, and $\chi_0 \to +\infty$ so that $\rho_{M0}\delta_0 = M_0 =$ constant ,and $\delta_0\chi_0 = \chi_{00} =$ constant, we obtain the delta function case $\chi(x) = \chi_{00}\delta(x)$.

The Delta Function Case

The equations in this case are

$$M_0 \ddot{q}(t) = -\frac{1}{8\pi}\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\Bigg|_{x=q(t)-}^{x=q(t)+},$$

$$\frac{\partial^2 A_0}{\partial x^2}(x,t) = \frac{1+4\pi\chi_{00}\delta[x-q(t)]}{c^2}\frac{\partial^2 A_0}{\partial t^2}(x,t).$$

We studied this model in L.O. Castaños, R. Weder Phys. Rev A. **89** (2014) 063807, in the half-line, $x > 0$, with a fixed perfect mirror at $x = 0$, $A_0(0,t) = 0$.

The delta-function model with both mirrors fixed, i.e. $q(t)$ constant has been studied by several authors, for example, by

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A Lagrangian

We derive a Lagrangian density for the field in the case where the conductivity is zero, that is, $\sigma(x'') = 0$.

Since the mirror is (approximately) at rest in $\mathbf{MS}_0$ during the time interval $[-t_1'', t_1'']$, it follows that the Lagrangian density for the electromagnetic field in $\mathbf{MS}_0$ is given by

$$\begin{aligned}\mathcal{L}'' &= \frac{1}{8\pi}\Big[\mathbf{E}''(x'', t'') \cdot \mathbf{D}''(x'', t'') \\ &\qquad - \mathbf{B}''(x'', t'') \cdot \mathbf{H}''(x'', t'')\Big] , \\ &= \frac{1}{8\pi}\left[\epsilon(x'')\mathbf{E}''(x'', t'')^2 - \mathbf{B}''(x'', t'')^2\right] ,\end{aligned}$$

for $t'' \in [-t_1'', t_1'']$ and the dielectric function

$$\epsilon(x'') = 1 + 4\pi\chi(x'') .$$

Using the transformation rules of the electric and the magnetic field we obtain the Lagrangian density in the laboratory system **LS**,

$$\mathcal{L} = \frac{1}{8\pi} \begin{pmatrix} \mathbf{E}(x,t) \\ \mathbf{B}(x,t) \end{pmatrix}^T \left(\mathbb{M}_0^{-1}\right)^T \begin{pmatrix} \epsilon(x'')\mathbb{I}_3 & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & -\mathbb{I}_3 \end{pmatrix} \times \\ \times \mathbb{M}_0^{-1} \begin{pmatrix} \mathbf{E}(x,t) \\ \mathbf{B}(x,t) \end{pmatrix},$$

with $x'' = \gamma(t)[x - q(t)]$ and $\mathbb{A}^T$ the transpose of matrix $\mathbb{A}$.

This Lagrangian density is valid for an arbitrary electromagnetic field. For the special case of transverse fields

$$\mathbf{A}(x,t) = A_0(x,t)\,\mathbf{z},\ V(x,t) = 0,$$

$$\mathbf{B}(x,t) = \nabla \times \mathbf{A}(x,t) = -\frac{\partial A_0}{\partial x}(x,t)\mathbf{y},$$

$$\mathbf{E}(x,t) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}(x,t) = -\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\mathbf{z}$$

we obtain the following Lagrangian density:

$$\mathcal{L} = \frac{1}{8\pi}\left\{\left[\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\right]^2 - \left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\right\} + \frac{\gamma(t)^2\chi_{\rm LS}(x,t)}{2}\left[\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t) + \beta(t)\frac{\partial A_0}{\partial x}(x,t)\right]^2. \tag{2}$$

Observe that (2) expresses the Lagrangian density as the sum of a part corresponding to the free field plus a part associated with the presence of the mirror.

This Lagrangian density (2) is identical to the Lagrangian density given in Cheung H K and Law C K, *Phys. Rev. A* **84**, (2011) 023812 (see also references therein), in the case where $\chi(x) = \chi_0$ for, $-\delta_0/2 \leq x \leq \delta_0/2$ and zero elsewhere.
The Euler-Lagrange equation associated with the Lagrangian density (2) is then given by

$$\frac{\partial}{\partial t}\left\{\frac{\partial \mathcal{L}}{\partial\left[\partial_t A_0(x,t)\right]}\right\} + \frac{\partial}{\partial x}\left\{\frac{\partial \mathcal{L}}{\partial\left[\partial_x A_0(x,t)\right]}\right\} = 0\,, \tag{3}$$

where ∂_t and ∂_x denote partial derivatives with respect to t and x, respectively. One can show from (2) that the Euler-Lagrange equation (3) does indeed give the equation that we obtained for $A_0(x,t)$ with $\sigma_{\text{LS}}(x,t) = 0$.

One can be interested in using a simpler Lagrangian density from which approximate equations for the field can be obtained (for example, to first order in the velocity and the acceleration of the mirror). One way to achieve this is to expand the Lagrangian density in (2) in powers of $\beta(t)$ as follows:

$$\mathcal{L} \;=\; \mathcal{L}_0 + \beta(t)\mathcal{L}_1 + \frac{1}{2}\beta(t)^2\mathcal{L}_2 \;+ \dots\,, \tag{4}$$

where the first three terms are given by

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{8\pi}\left\{\epsilon[x-q(t)]\left[\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\right]^2 - \left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\right\},\\
\mathcal{L}_1 &= \chi[x-q(t)]\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\frac{\partial A_0}{\partial x}(x,t)\,,\\
\mathcal{L}_2 &= \chi[x-q(t)]\left\{\left[\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\right]^2 + \left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\right\}\\
&\quad + \frac{x-q(t)}{2}\frac{d\chi}{dx''}[x-q(t)]\left[\frac{1}{c}\frac{\partial A_0}{\partial t}(x,t)\right]^2 \,.
\end{aligned} \tag{5}$$

The term $\beta(t)^n \mathcal{L}_n/n!$ introduces terms of order higher than n in the Euler-Lagrange equations for the field, since one has to calculate derivatives with respect to t and factors $\chi[x - q(t)]$ are present. Therefore, the equations of order n in $\dot{q}(t)$ and $\ddot{q}(t)$ for the field cannot be deduced exactly with (4) if one neglects $\mathcal{L}_m$ for $m > n$, since terms of order $n + 1$ would have to be neglected in the Euler-Lagrange equations to obtain the correct equations of motion. This shows that it appears not to be possible to have an approximate Lagrangian density of a given order for the field that yields the correct equations without having to discard terms.

The argument above holds for a continuously differentiable electric susceptibility $\chi(x'')$. For the piecewise constant $\chi(x'')$ the derivatives of $\chi[x - q(t)]$ are zero inside and outside of the mirror, so that one obtains the correct equations of order n in $\dot{q}(t)$ and $\ddot{q}(t)$ for the field if one neglects $\mathcal{L}_m$ for $m > n$.

For an approximate Lagrangian of order one in $\beta(t)$ for the complete mirror+field system, with $\chi(x) = \chi_0$ for, $-\delta_0/2 \leq x \leq \delta_0/2$ and zero elsewhere, we refer to Cheung H K and Law C K, *Phys. Rev. A* **84**, (2011) 023812 , where the following Lagrangian is proposed:

$$L^{(1)} = \frac{1}{2}M_0\dot{q}(t)^2 + V[q(t)] + \int_0^L dx\,[\mathcal{L}_0 + \beta(t)\mathcal{L}_1]\,.$$

Here $V[q(t)]$ is a potential affecting the mirror and $[0, L]$ is the region where the mirror can move. Moreover, two perfect, fixed mirrors are located at $x = 0$ and $x = L$. Cheung and Law used their Lagrangian quantize the mirror+field system .
This Lagrangian gives the correct equation for the field to order one in $\dot{q}(t)$ and $\ddot{q}(t)$ but it only gives the correct force on the mirror to order zero in $\dot{q}(t)$ and $\ddot{q}(t)$. In this sense it is not consistent. Furthermore, it does not recover the correct time-dependent mass and velocity-dependent force affecting the mirror.

Back to the Delta Function Model

Recall that the equations in this case are

$$M_0\ddot{q}(t) = -\frac{1}{8\pi}\left[\frac{\partial A_0}{\partial x}(x,t)\right]^2\Bigg|_{x=q(t)-}^{x=q(t)+},$$

$$\frac{\partial^2 A_0}{\partial x^2}(x,t) = \frac{1+4\pi\chi_0\delta[x-q(t)]}{c^2}\frac{\partial^2 A_0}{\partial t^2}(x,t).$$

We studied this model in L.O. Castaños, R. Weder Phys. Rev A. **89** (2014) 063807, in the half-line, $x>0$, with a fixed perfect mirror at $x=0$, $A_0(0,t)=0$.

In the case where the mirrors are fixed, i.e. $q(t)$ constant, the scattering solutions to the field equation are given by (B. Baseia and H. M. Nussenzveig, 1984),

$$A_{0,k}(x,t) = \tilde{V}_k[x, q(t)] e^{-i\omega t} \ ,$$

$$\tilde{V}_k[x, q(t)] = \begin{cases} \frac{i}{2} L_k[q(t)] e^{i\{\delta_k[q(t)] - kq(t)\}} \left(e^{-ikx} - e^{ikx} \right) \\ \qquad \text{if } \ 0 \leq x \leq q(t), \\ \frac{i}{2}\sqrt{\frac{2}{\pi}} \left\{ e^{-ikx} - e^{ikx} e^{i2\{\delta_k[q(t)] - kq(t)\}} \right\} \\ \qquad \text{if } \ x > q(t). \end{cases}$$

For fixed $k > 0$, the function $L_k\left[q(t)\right]$ is maximized (minimized) for a discrete set of values q_{2n} (q_{2n+1}) of $q(t)$. If $4\pi\chi_0 k \gtrsim 5$, then one has to good approximation

$$\begin{aligned} kq_{2n} &\simeq n\pi + \frac{1}{4\pi\chi_0 k}\,, \\ kq_{2n+1} &\simeq \left(n + \frac{1}{2}\right)\pi + \frac{1}{4\pi\chi_0 k}\,, \end{aligned}$$

and

$$\begin{aligned} L_k(q_{2n}) &\simeq \sqrt{\frac{2}{\pi}}(4\pi\chi_0 k)\,, \\ L_k(q_{2n+1}) &\simeq \sqrt{\frac{2}{\pi}}\left(\frac{1}{4\pi\chi_0 k}\right)\,. \end{aligned} \tag{6}$$

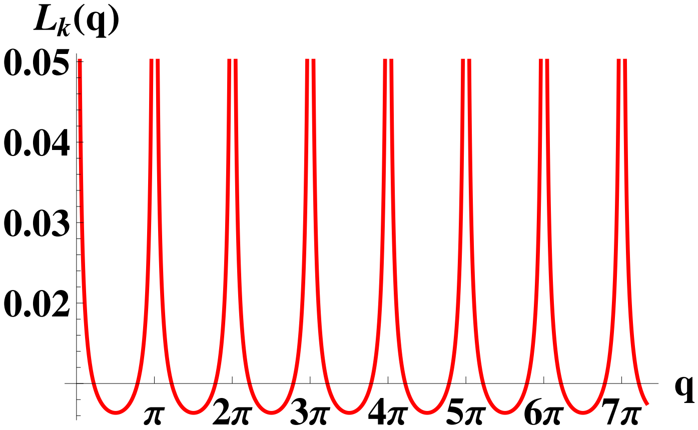

Figure: The figure illustrates $L_k(q)$ as a function of q for $\chi_0 = 10$ m and $k = 1$ m^{-1}. For a better illustration only part of the y-axis is shown.

The maxima correspond to the resonances (scattering frequencies) of the delta function.

When the mirror is close to a position where the field is in resonance, the field is big inside the cavity and small outside and the radiation force pushes the mirror to the right.

On the contrary, when it is far from a position where the field is in resonance, the field is small inside the cavity and big outside and the radiation force pushes the mirror to the left.

This gives rise to a rich dynamics for the mirror on spite of the simplicity of the model.

We assume that the field has only one mode with $k = k_N^0$ (this is possible at leading order)

$$A_0(x,t) = \mathrm{Re}\left\{ \, 2g_0 \, \tilde{V}_{k_N^0}[x, q(t)] e^{-i\omega_0 t} \, \right\} \quad \omega_0 = c \, k_N^0,$$

and g_0 a coupling constant. We denote,

$$\xi = 4\pi\chi_0 k_N^0.$$

The equation for the mirror is:

$$M_0 \ddot{q}(t) = \left(\frac{g_0\omega_0}{\pi c}\right)^2 f_{\mathrm{RWA}}\left[k_N^0 q(t)\right] \times$$

$$\left\{ 1 + \cos\left[2\omega_0 t + 2k_N^0 q(t) - 2\delta_{k_N^0}[q(t)] \,\right] \,\right\}$$

where,

$$f_{\mathrm{RWA}}(x) \quad = \quad -\frac{1}{2}\left[1 - \frac{1}{1 + \xi^2 \sin^2(x) - \xi \sin(2x)}\right]$$

The force $f_{\text{RWA}}(x)$ can be derived from a potential $V_{\text{RWA}}(x)$:

$$f_{\text{RWA}}(x) = -\frac{d}{dx} V_{\text{RWA}}(x) .$$

If $(2m-1)\pi/2 \leq x \leq (2m+1)\pi/2$ and $x \geq 0$ with $m \in \mathbb{Z}^+$, then $V_{\text{RWA}}(x)$ is given by

$$V_{\text{RWA}}(x) = \frac{x}{2} - \frac{1}{2}\tan^{-1}\left[(1+\xi^2)\tan(x) - \xi \right] - \frac{1}{2}\left[\tan^{-1}(\xi) + m\pi \right] . \tag{7}$$

$V_{\text{RWA}}(x)$ is a periodic function of period π. Then, $V_{\text{RWA}}(k_N^0 x)$ has period half the wavelength.

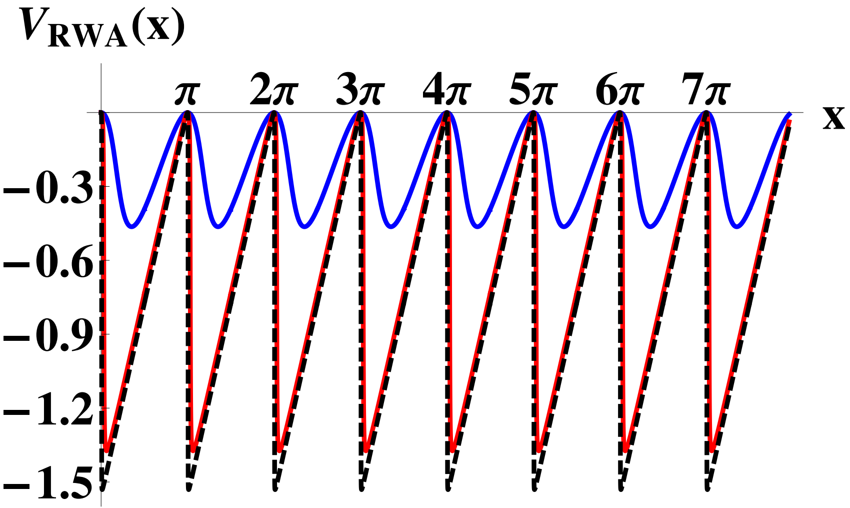

Figure: The figure shows $V_{\rm RWA}(x)$ for $\xi = 1$ (blue-solid line), 10 (red-solid line), and 50 (black-dashed line).

Nondimensional Variables

Define

$$\Delta = \sqrt{\frac{g_0^2\omega_0^3}{\pi^2 M_0 c^3}}\,, \quad \xi = 4\pi\chi_0 k_N^0,$$

$$\Omega = 2\frac{\omega_0}{\Delta}\,, \quad x(\tau) = k_N^0 q\left(\frac{\tau}{\Delta}\right).$$

Δ has units of 1/s, $\tau = \Delta t$ is the non-dimensional time.
ξ, Ω, and $x(\tau)$ are non-dimensional quantities.
$1/k_N^0$ (1 over the wavenumber of the field) is the characteristic length of the system.
$1/\Delta$ is the characteristic time. It can be thought of as 1 over the time-scale in which the position of the movable mirror changes appreciably.
$\Omega = (4\pi/\Delta)/(2\pi/\omega_0)$ is interpreted to be the time-scale $4\pi/\Delta$ in which the movable mirror changes appreciably divided by the time-scale $2\pi/\omega_0$ in which the field changes appreciably.

Nondimensional Mirror's Equation

$$\frac{d^2x}{d\tau^2}(\tau) = \left\{1 + \cos\left[\Omega\tau + 2x(\tau) - 2\delta_{k_N^0}\left(\frac{x(\tau)}{k_N^0}\right)\right]\right\} f_{\text{RWA}}\left[x(\tau)\right].$$

There Are Three Regimes

Low Field Intensity or Rotating Wave Approximation Regime

$$\Omega >> 1.$$

In this case the cos term oscillates rapidly in time, and it can be averaged to zero.

$$\frac{d^2 x}{d\tau^2}(\tau) = f_{\text{RWA}}(x) = -\frac{d}{dx} V_{\text{RWA}}(x).$$

The energy of the mirror is conserved

$$E\left[x(\tau), x'(\tau)\right] = \frac{1}{2}\left[x'(\tau)\right]^2 + V_{\text{RWA}}\left[x(\tau)\right] \; .$$

For negative energies the mirror oscillates in a well of the potential. For positive energies it can travel far away.

High and Intermediate Field Intensity Regimes

High Field Intensity Regime.

$$\Omega << 1$$

It is not compatible with our model.

Intermediate Field Intensity Regime.
Between the low- and high- intensity regimes.
It is compatible with our model
The force is time dependent and the energy of the mirror is not conserved. The mirror can jump from one well to another.

We also considered the cases where there is friction and a harmonic oscillator potential

Note that
We considered the exact modes of the field.
The mirror can deviate far from an equilibrium position.
In the RWA regime, for negative energies, the mirror performs oscillations around one of the minimizers of the potential.

Approximation of the Potential

$$V_{\text{RWA}}^{\text{approx}}(x) \equiv \begin{cases} 0 & \text{if} \quad n\pi \le x \le n\pi + \frac{1}{\xi}, \\ V_{\text{RWA}}\left(\frac{2}{\xi}\right) & \\ & \text{if} \quad n\pi + \frac{1}{\xi} < x \le n\pi + \frac{2}{\xi}, \\ V_{\text{RWA}}\left(\frac{2}{\xi}\right) + m_+ \left[x - \left(n\pi + \frac{2}{\xi} \right) \right] & \\ & \text{if} \quad n\pi + \frac{2}{\xi} \le x \le (n+1)\pi . \end{cases} \tag{8}$$

for $n\pi \le x \le (n+1)\pi, \ n \in \mathbb{Z}^+$. Here

$$m_+ = -\frac{V_{\text{RWA}}(2/\xi)}{\pi - 2/\xi} \simeq \frac{1}{2} . \tag{9}$$

We note that the point $x = (n\pi + 1/\xi)$ corresponds (approximately) to a maximizer x_{2n} of $f_{\text{RWA}}(x)$.

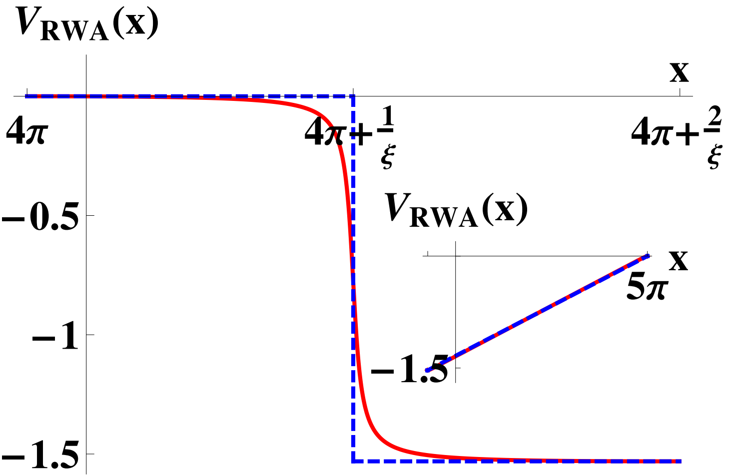

Figure: It shows a close up of one of the wells of $V_{\rm RWA}(x)$ (red-solid line) and the approximate $V_{\rm RWA}^{\rm approx}(x)$ (blue-dashed line) for $\xi = 50$ and the region $[4\pi, 4\pi + 2\xi^{-1}]$. The inside figure shows the region $[4\pi + 2\xi^{-1}, 5\pi]$

Discussion of the Dynamics. The Low Intensity Regime

In the low field intensity regime and for $\xi >> 1$ we have the following approximate solution for negative energy. Assuming that the moving mirror is in the *n*-th well one gets that

$$x_{\mathrm{RWA}}(\tau) = \\ = \begin{cases} \left(n\pi + \frac{2}{\xi}\right) + v_0(\tau - \tau_{2k}) - \frac{m_+}{2}(\tau - \tau_{2k})^2 \\ \qquad\qquad \text{if} \ \ \tau_{2k} \leq \tau \leq \tau_{2k+1} \ , \\ \left(n\pi + \frac{2}{\xi}\right) - v_0(\tau - \tau_{2k+1}) \\ \qquad\qquad \text{if} \ \ \tau_{2k+1} \leq \tau \leq \tau'_{2k+1} \ , \\ \left(n\pi + \frac{1}{\xi}\right) + v_0(\tau - \tau'_{2k+1}) \\ \qquad\qquad \text{if} \ \ \tau'_{2k+1} \leq \tau \leq \tau_{2(k+1)} \ . \end{cases}$$

Here $k \in \mathbb{Z}$ and τ_0 is an instant such that $x(\tau_0) = n\pi + 2/\xi$ (a minimum of the potential) and $v_0 = (dx/d\tau)(\tau_0) > 0$.

For $k \in \mathbb{Z}$ one has

$$\tau_{2k+1} - \tau_{2k} = \frac{2v_0}{m_+},$$
$$\tau_{2(k+1)} - \tau'_{2k+1} = \tau'_{2k+1} - \tau_{2k+1} = \frac{1}{\xi v_0}.$$

Observe that $x(\tau_{2k}) = n\pi + 2/\xi$ and $v_0 = (dx/d\tau)(\tau_{2k})$ for all $k \in \mathbb{Z}$. Furthermore,

$$m_+ = -\frac{V_{\mathrm{RWA}}(2/\xi)}{\pi - 2/\xi}.$$

The period of the bounded motion is given by

$$P = \frac{2v_0}{m_+} + \frac{2}{\xi v_0}.$$

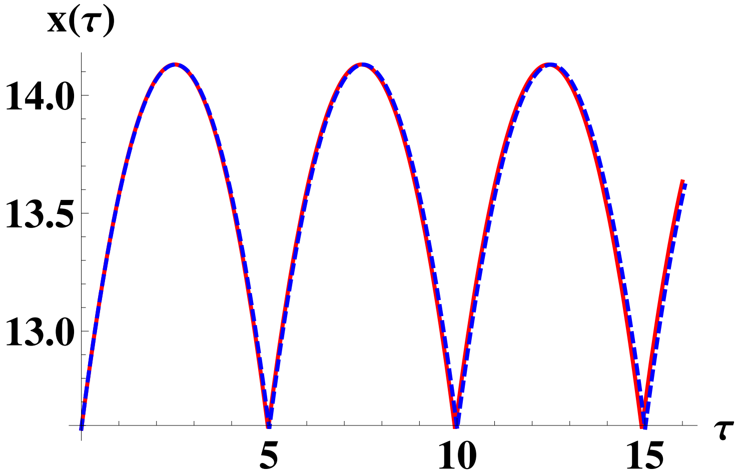

Figure: The figure illustrates the solution (red-solid line) $x_{\mathrm{RWA}}(\tau)$ computed numerically for $\xi = 50$ and the initial conditions $x(0) = 4\pi + 1/\xi$, $x'(0) = 0$. It also shows the approximate solution(blue-dashed line) corresponding to the same energy.

Discussion of the Dynamics. The Intermediate Intensity Regime

Recall that in this case the mirror's equation is given by

$$\frac{d^2x}{d\tau^2}(\tau) = \left\{1 + \cos\left[\Omega\tau + 2x(\tau) - 2\delta_{k_N^0}\left(\frac{x(\tau)}{k_N^0}\right)\right]\right\} f_{\text{RWA}}\left[x(\tau)\right].$$

Since the force depends explicitly on time energy is not conserved and the dynamics is more complicated.

If the movable mirror is initially confined to one of the wells of $V_{\text{RWA}}(x)$, at future times it may jump out of the well and then be confined for some time in another well. Also, oscillations in one well similar to those obtained with the RWA are possible for large enough Ω. The next Figure illustrates these types of behaviors for several values of Ω and compares them with the solution in the RWA (black-dotted line) calculated numerically . Notice how $x(\tau)$ tends to the solution in the RWA for increasing Ω.

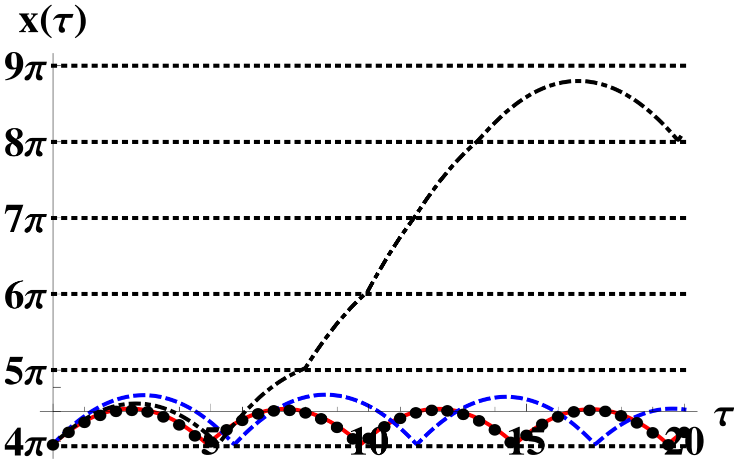

Figure: $x(\tau)$ with $\Omega = 100$ (black-dotdashed line), 250 (blue-dashed line), and 500 (red-solid line) for $\xi = 10$ and the initial conditions $x(0) = 4\pi + 1/\xi$ and $x'(0) = 0$. The horizontal black-dotted lines indicate the beginning and the ends of the potential wells of $V_{\text{RWA}}(x)$. The figure also illustrates the solution in the RWA (black-dotted line).

Adding Friction in the Low Intensity Regime.

The equation for the mirror is now:

$$\frac{d^2x}{d\tau^2}(\tau) + \Gamma\frac{dx}{d\tau}(\tau) \;=\; f_{\text{RWA}}\left[x(\tau)\right] \;.$$

We have the following approximate solution for $\xi \gg 1$ and $E[x(0), x'(0)] < 0$.

$$\begin{aligned} x_{\text{RWA}}(\tau) \;&=\; c_2(\tau_0)e^{-\Gamma(\tau-\tau_0)} - \frac{m_+}{\Gamma}(\tau-\tau_0) \\ &\quad + c_1(\tau_0) + \frac{m_+}{\Gamma^2}\left[1 - e^{-\Gamma(\tau-\tau_0)}\right] \;, \end{aligned}$$

with

$$\begin{aligned} c_1(\tau_0) \;&=\; x_{\text{RWA}}(\tau_0) + \frac{1}{\Gamma}x'_{\text{RWA}}(\tau_0) \;, \\ c_2(\tau_0) \;&=\; -\frac{1}{\Gamma}x'_{\text{RWA}}(\tau_0) \;, \end{aligned}$$

if $(n\pi + 2/\xi) \leq x_{\text{RWA}}(\tau) \leq (n+1)\pi$ for $\tau_0 \leq \tau \leq \tau_1$ and some $n \in \mathbb{Z}^+$.

$$x_{\mathrm{RWA}}(\tau) \quad = \quad x_{\mathrm{RWA}}(\tau_1) + \frac{x'_{\mathrm{RWA}}(\tau_1)}{\Gamma}\left[1 - e^{-\Gamma(\tau-\tau_1)}\right] \ ,$$

if $(n\pi + 1/\xi) \leq x_{\mathrm{RWA}}(\tau) \leq (n\pi + 2/\xi)$ for $\tau_1 \leq \tau \leq \tau_2$.

In order to get the complete trajectory, the solutions have to be pasted together. Moreover, it has to be taken into account that $x_{\mathrm{RWA}}(\tau)$ bounces elastically from a potential wall if it reaches $(n\pi + 1/\xi)$.

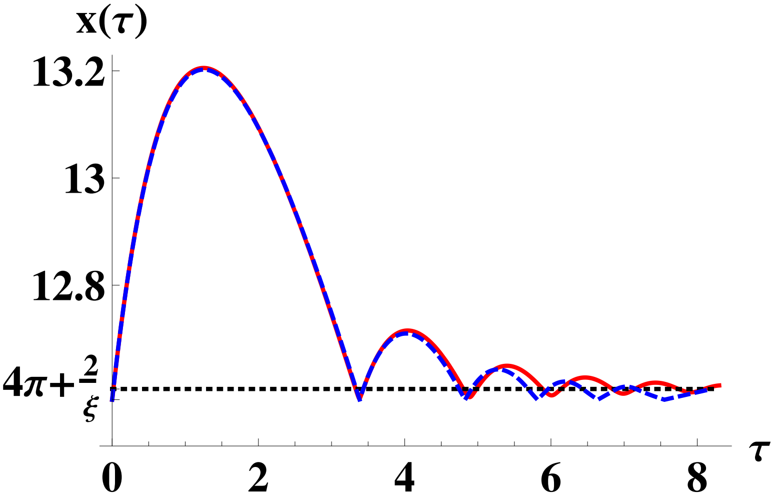

Figure: Comparison of the The numerical solution (red-solid line) and the approximate $x_{\text{RWA}}(\tau)$ (blue-dashed line) for $\xi = 50$, $\Gamma = 1$, and the initial conditions $x(0) = (4\pi + 1/\xi)$ and $x'(0) = 0$.

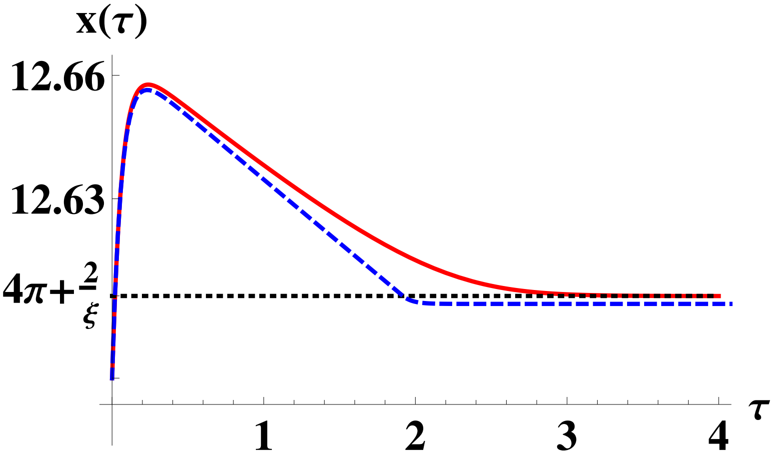

Figure: Comparison of the The numerical solution (red-solid line) and the approximate $x_{\mathrm{RWA}}(\tau)$ (blue-dashed line) for $\xi = 50$, $\Gamma = 16$, and the initial conditions $x(0) = (4\pi + 1/\xi)$ and $x'(0) = 0$.

The differences between the analytical and the numerical solutions are due to two facts.
First, the approximation does not take into account the curvature of the potential $V_{\mathrm{RWA}}(x)$.
Second, for large enough times the movable mirror spends more time near the minimizer x_n^{**} of $V_{\mathrm{RWA}}(x)$ where the curvature is important.

Nevertheless, $x_{\text{RWA}}(\tau)$ allows one to understand the behavior of the mirror.

The movable mirror approximately behaves like a particle in free fall subject to friction (linear in the velocity) when it is in the region between the minimizer $x_n^{**} \simeq (n\pi + 2/\xi)$ and the maximizer $x_{n+1}^{*} = (n+1)\pi$ of $V_{\text{RWA}}(x)$.

On the other hand, it behaves approximately like a particle subject only to friction in the region between the maximizer $x_{2n} \simeq (n\pi + 1/\xi)$ of $f_{\text{RWA}}(x)$ and x_n^{**}.

Moreover, the mirror bounces elastically from an impenetrable potential wall if it reaches x_{2n}, which corresponds to the maximizer of $f_{\text{RWA}}(x)$.

Adding Friction in the Intermediate Regime

The equation for the mirror is:

$$\frac{d^2x}{d\tau^2}(\tau) + \Gamma \frac{dx}{d\tau}(\tau)$$
$$= \left\{ 1 + \cos\left[\Omega\tau + 2x(\tau) - 2\delta_{k_N^0}\left(\frac{x(\tau)}{k_N^0}\right)\right]\right\} \times$$
$$\times f_{\text{RWA}}\left[x(\tau)\right] \ .$$

The numerical results show that depending on the values of Ω, Γ, and ξ, the mirror can simply tend to a minimizer of $V_{\text{RWA}}(x)$ in a similar way as before or, if the friction is small enough, it can jump between potential wells before finally settling in one of the minimizers. $x(\tau)$ was computed numerically from for several values of ξ, Γ, and Ω and for initial conditions such that the mirror starts from rest at a position where ω_0 coincides with one of the cavity resonance frequencies.

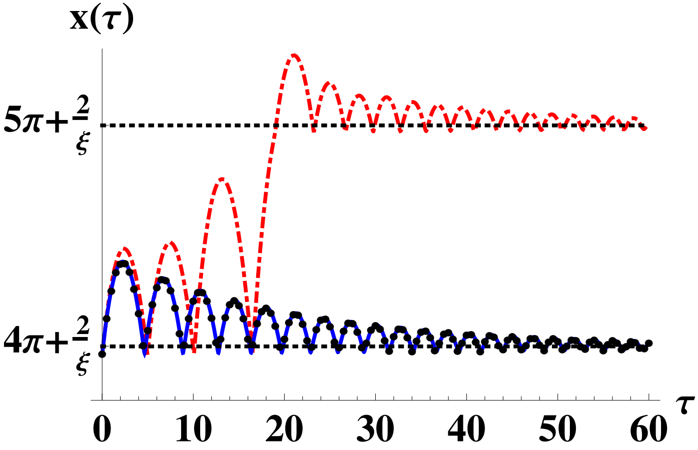

Figure: $\xi = 10$, $\Gamma = 0.07$, $\Omega = 100$ (blue-solid line), $\Omega = 500$ (red-dotdashed line). Initial conditions $x(0) = 4\pi + 1/\xi$ and $x'(0) = 0$. It also shows $x_{\mathrm{RWA}}(\tau)$ (black-dotted line). The horizontal black-dotted lines $(4\pi + 2/\xi)$ and $(5\pi + 2/\xi)$ show approximately the position of minimizers x_n^{**} of $V_{\mathrm{RWA}}(x)$.

Thanks for your attention