Right and Left Electromagnetic Fields. Theory and Applications.

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In honor of Patrick Joly 60th Birthday

"La vieillesse est un naufrage"
(Général de Gaulle).

Right/Left Electromagnetic Fields

Unstationnary Maxwell Equations...

$$\begin{cases} ik\eta_0^{-1}E(x) - \mathrm{curl}H(x) = 0 \\ ik\eta_0 H(x) + \mathrm{curl}E(x) = 0 \end{cases}$$

- k wavenumber $= 2\pi F\sqrt{\epsilon_0\mu_0}$
- η_0 impedance of the vacuum $= \sqrt{\mu_0\epsilon_0^{-1}}$

$$\begin{cases} \text{Left EM field:} \quad eh^{+}(x) = E(x) + i\eta_0 H(x), \\ \text{Right EM field:} \quad eh^{-}(x) = E(x) - i\eta_0 H(x), \end{cases}$$

Equations Satisfied by $eh^{\pm}(x)$??

Right/Left Electromagnetic Fields

$$\begin{cases} (\times[-i]) \ : \ ikE - \mathrm{curl}\eta_0 H \ = \ 0 \\ + \text{ or } - \ \ : \ ik\eta_0 H + \mathrm{curl}E \ = \ 0 \end{cases}$$

$$\ldots\Rightarrow\ldots$$

$$k[E + i\eta_0 H] + \mathrm{curl}[E + i\eta_0 H] = 0$$

$$k[E - i\eta_0 H] - \mathrm{curl}[E - i\eta_0 H] = 0$$

...⇒...2 uncoupled equations

$$k eh^{\pm} \pm \mathrm{curl} eh^{\pm} = 0$$

An EM field will be said to be Left if its right component is 0:

$$(\text{i.e. } E(x) = i\eta_0 H(x))$$

Ex: Circular Polarized plane waves

Let $(\hat{x}, \hat{\theta}, \hat{\varphi})$ orthonormal, direct

$$\begin{cases} E_i^{\pm}(y) \; = \; e_0\,(\hat{\theta} \pm i\hat{\varphi})\, e^{ik\hat{x}\cdot y} \\ H_i^{\pm}(y) \; = \; \dfrac{-1}{ik\eta_0}\mathrm{curl} E_i^{\pm}(y) \end{cases}$$

$$i\eta_0 H_i^{\pm}(y) = -\hat{x} \times [\hat{\theta} \pm i\hat{\varphi}]\, e^{ik\hat{x}\cdot y}$$

...

$$\pm i\eta_0 H_i^{\pm}(y) = E_i^{\pm}(y) \quad \text{(circular polariz.)}$$

with $\pm = +$ we obtain a Left Field.

Ex: Right/Left Dipoles

p some vector,...

$$\begin{cases} ik\eta_0^{-1}E_1(x) - \mathrm{curl} H_1(x) &= \frac{1}{2}p\,\delta(x) \\ ik\eta_0 H_1(x) + \mathrm{curl} E_1(x) &= 0 \end{cases}$$

$$\begin{cases} ikE_2(x) - \mathrm{curl}\eta_0 H_2(x) &= 0 \\ ik\eta_0 H_2(x) + \mathrm{curl} E_2(x) &= \frac{1}{2}\eta_0 p(x) \end{cases}$$

$$(E(x), H(x)) = (E_2(x), H_2(x)) + i(E_1(x), H_1(x))$$

is a "Right dipole" with polarization vector p

$$k eh^+ - \mathrm{curl}\, eh^+ = p, \;\; k eh^- + \mathrm{curl}\, eh^- = 0$$

Imp. Bound. Cond. adapted to the vacuum

The impedance boundary condition

$$n(x) \times [E(x) \times n(x)] + \eta H(x) \times n(x) = 0, \; x \in \Gamma$$

is said to be adapted to the vacum iff

$$\eta = \eta_0$$

This condition

- is involved in the Silver-Müller radiating condition
- is the 1^{th} order absorbing boundary condition
- plays an important role in Domain Decomposition Method
- is an ideal condition to control the quality of a measurement chamber

Parenthesis: the $\Pi^{\pm}(\hat{x})$ projectors

$\hat{x}$ some unit vector, define

$$\Pi^{\pm}(\hat{x})v = \frac{1}{2}\Big(\hat{x} \times [v \times \hat{x}] \pm i\hat{x} \times v\Big)$$

$$\Big(\Pi^{\pm}(\hat{x})\Big)^{*} = \Pi^{\pm}(\hat{x}), \quad \Pi^{\pm}(\hat{x})^2 = \Pi^{\pm}(\hat{x})$$

$$\Pi^{+}(\hat{x})\Pi^{-}(\hat{x}) = \Pi^{-}(\hat{x})\Pi^{+}(\hat{x}) = 0$$

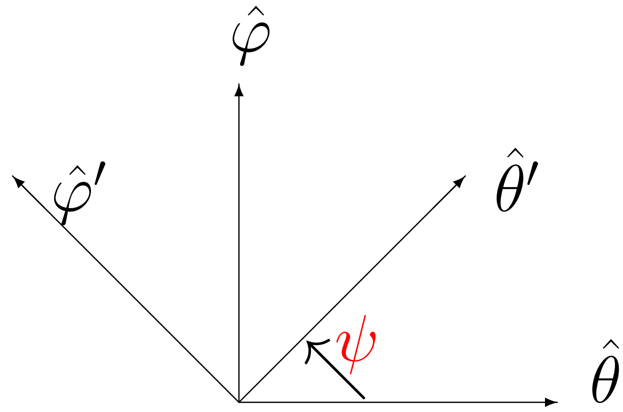
For any $(\hat{x}, \hat{\theta}, \hat{\varphi})$ orthonormal direct

$$\Pi^{\pm}(\hat{x})v \qquad = [v \cdot \overline{(\hat{\theta} \pm i\hat{\varphi})}]\,(\hat{\theta} \pm i\hat{\varphi})/2$$

$$\Pi^{\pm}(\hat{x})[\hat{\theta} \mp i\hat{\varphi}] = 0, \quad \Pi^{\pm}(\hat{x})\hat{x} = 0$$

Two rank one orthogonal projectors

$$\begin{aligned}\Pi^{\pm}(\hat{x})v &= \left[v \cdot \overline{(\hat{\theta} \pm i\hat{\varphi})}\right](\hat{\theta} \pm i\hat{\varphi}) \\ &= \left[v \cdot \overline{(\hat{\theta}' \pm i\hat{\varphi}')}\right](\hat{\theta}' \pm i\hat{\varphi}')???\end{aligned}$$



$$\hat{\theta}' \pm i\hat{\varphi}' = e^{-i\psi}(\hat{\theta} \pm i\hat{\varphi})$$

ψ the angle of the curlation from $(\hat{\theta}', \hat{\varphi}')$ to $(\hat{\theta}, \hat{\varphi})$

The $\Pi^{\pm}(\hat{x})$'s are "intrinsic".

Back to the IBC

$$n(x) \times [E(x) \times n(x)] \quad - \; n(x) \times \eta_0 H(x) = 0$$

$$\text{make } n(x)\times \Rightarrow$$

$$n(x) \times [\eta_0 H(x) \times n(x)] \; + \; n(x) \times E(x) = 0$$

$$n \times [E(x) + i\eta_0 H(x)] \times n + in \times [E(x) + i\eta_0 H(x)]$$

$$n \times [E(x) - i\eta_0 H(x)] \times n - in \times [E(x) - i\eta_0 H(x)]$$

$$\begin{cases} k\,eh^{\pm}(x) \pm \operatorname{curl} eh^{\pm}(x) = 0, x \in \Omega^e \\ \Pi^{\pm}(n) eh^{\pm}(x) = 0, \quad x \in \Gamma \\ \lim_{|x| \to \infty} |x| \Pi^{\mp}(\hat{x}) eh^{\pm}(x) = 0 \text{(radiating condition)} \end{cases}$$

Uniqueness

$$ku - \mathrm{curl} u = 0, \quad (k^2 u - \mathrm{curl}\mathrm{curl} u = 0)$$

Stoke Theorem in Ω_R...

$$\Im \int_{\partial\Omega_R} u(x) \cdot [\overline{u}(x) \times \nu(x)] = 0$$

from what we deduce

$$\int_\Gamma |\Pi^+(n)u|^2 + \int_{S_R} |\Pi^-(\hat{x})u|^2 =$$

$$\int_\Gamma |\Pi^-(n)u|^2 + \int_{S_R} |\Pi^+(\hat{x})u|^2.$$

$$u_{|\Gamma}(x) \times n(x) = 0, \lim_{|x|\to\infty} \int_{S_R} |u(x) \times \hat{x}|^2 = 0$$

Conservation of the circular polarization.

Theorem Send a Right circular polarized plane wave on a scatterer with IBC adapted to the vacuum creates a Right electromagnetic scattered field.

$$E_i(x) = i\eta_0 H_i(x) \Rightarrow E_d(x) = i\eta_0 H_d$$

.

Theorem Send a Right dipole on a scatterer with IBC adapted to the vacuum creates a Right electromagnetic field.

$$E(x) = i\eta_0 H(x)$$

.

Application to the validation of EM codes.

If you have a code to compute the radiating EM field

$$\begin{cases} i\omega\underline{\underline{\epsilon}}(x)E(x) - \mathrm{curl}H(x) = J(x) \\ i\omega\underline{\underline{\mu}}(x)H(x) + \mathrm{curl}E(x) = M(x) \\ n(x) \times [E_{|_\Gamma}(x) \times n(x)] + \eta(x)n(x) \times H_{|_\Gamma}(x) = 0 \end{cases}$$

then take

$$\eta(x) = \eta_0, \quad \underline{\underline{\mu}}(x) = \eta_0^2\,\underline{\underline{\epsilon}}(x)$$

(medium adapted to the vacuum)

$$\text{impose } M(x) = i\eta_0 J(x)$$

you must have

$$E_h(x) \simeq i\eta_0 H_h(x).$$

The Weston Theorem.

Send a plane wave with direction along $\hat{z}$, the axis of revolution of some axisymmetric scatterer with IBC adapted to the vacuum;

Observe the far field at $-\hat{z}$:
you obtain 0!

The proof of this not trivial result is quasi immediate with the previous theoritical tool.

Hint: choose the polarization to be left; then the scattered field is left. Conclude with the formula that gives the far field: the integration over the azymuth is $\int_0^{2\pi} e^{2i\varphi} d\varphi = 0$.

Classical Potentials

Let $\Omega^e = \mathbb{R}^3/\Omega^i$, Ω^i bounded

Let $(E(x), H(x))$ be a radiating Elec.Field in Ω^e; Define

$$J(y) = n(y) \times H(y), \qquad M(y) = n(y) \times E(y)$$

with $n(y)$ normal pointing into the exterior of Ω_i; then

$$\begin{cases} E(x) & = \dfrac{-\eta_0}{ik} \operatorname{curl}\operatorname{curl} \ SJ(x) + \ \operatorname{curl} \ SM(x) \\ H(x) & = \dfrac{+1}{ik\eta_0} \operatorname{curl}\operatorname{curl} \ SM(x) + \ \operatorname{curl} \ SJ(x) \end{cases}$$

with

$$SC(x) = \int_\Gamma \frac{e^{ik|x-y|}}{4\pi|x-y|} C(y) \, ds(y), \qquad \Gamma = \partial\Omega^e$$

Polarized Potentials/currents

Algebraic manipulations...

$$\begin{cases} E(x) &= \dfrac{1}{k}\,\text{curl}\,\text{curl}\; S i\eta_0 J(x) + \;\text{curl}\;\; SM(x) \\ i\eta_0 H(x) &= \dfrac{1}{k}\,\text{curl}\,\text{curl}\; SM(x) + \;\text{curl}\;\; S i\eta_0 J(x) \end{cases}$$

Define

$$\begin{cases} eh^{\pm}(x) &= E(x) \pm i\eta_0 H(x), \\ jm^{\pm}(y) &= n(y) \times [E(y) \pm i\eta_0 H(y)] \\ &= n(y) \times eh^{\pm}(y) \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} eh^{+}(x) &= \left[\dfrac{1}{k}\,\text{curl}\,\text{curl}\; + \;\text{curl}\right] S jm^{+}(x) \\ eh^{-}(x) &= \left[\dfrac{1}{k}\,\text{curl}\,\text{curl}\; - \;\text{curl}\right] S jm^{-}(x) \end{cases}$$

Classical Far fields

At infiniy

$$\begin{cases} E(x) \;=\; \dfrac{e^{ik|x|}}{|x|}\left(E_\infty(\hat{x}) + O(|x|^{-1})\right), \; \hat{x} = \dfrac{x}{|x|} \\ H(x) \;=\; \dfrac{e^{ik|x|}}{|x|}\left(H_\infty(\hat{x}) + O(|x|^{-1})\right) \end{cases}$$

with

$$\begin{cases} E_\infty(\hat{x}) \;=\; \dfrac{ik}{4\pi}\hat{x} \times \displaystyle\int_\Gamma [M(y) + \eta_0 J(y) \times \hat{x}] e^{-iky\cdot\hat{x}} \\ H_\infty(\hat{x}) \;=\; \dfrac{ik}{4\pi}\hat{x} \times \displaystyle\int_\Gamma [J(y) - \eta_0^{-1} M(y) \times \hat{x}] e^{-iky\cdot\hat{x}} \\ H_\infty(\hat{x}) \;=\; \eta_0^{-1}\, \hat{x} \times E_\infty(\hat{x}) \end{cases}$$

Polarized Far fields

$$eh^{\pm}(x) = \frac{e^{ik|x|}}{|x|}\left(eh_{\infty}{}^{\pm}(\hat{x}) + O(|x|^{-1}) \right)$$

$$eh_{\infty}{}^{\pm}(\hat{x}) = E_{\infty}(\hat{x}) \pm i\hat{x} \times E_{\infty}(\hat{x}) = 2\Pi^{\pm}(\hat{x}) E_{\infty}(\hat{x})$$

$$\begin{cases} eh_{\infty}{}^{+}(\hat{x}) \; = \; \dfrac{k}{2\pi} \displaystyle\int_{\Gamma} \Pi^{+}(\hat{x}) jm^{+}(y)\, e^{-iky\cdot\hat{x}} ds(y) \\ eh_{\infty}{}^{-}(\hat{x}) \; = \; \dfrac{k}{2\pi} \displaystyle\int_{\Gamma} \Pi^{-}(\hat{x}) jm^{-}(y)\, e^{-iky\cdot\hat{x}} ds(y) \end{cases}$$

or, for any $(\hat{x}, \hat{\theta}, \hat{\varphi})$ direct,

$$eh_{\infty}{}^{\pm}(\hat{x}) = (\hat{\theta} \pm i\hat{\varphi}) \frac{k}{4\pi} \underbrace{\int_{\Gamma} jm^{\pm}(y) \cdot \overline{(\hat{\theta} \pm i\hat{\varphi})} e^{-iky\cdot\hat{x}}}_{\textbf{Scalar}}$$

“Multipoles” Formulas I

c_s far away from c_t;

x_s close to c_s, x_t close to c_t, define

$$d = c_t - c_s, \quad v = (x_t - c_t) - (x_s - c_s), \quad [v[< |d|$$

then, we have...

$$\frac{e^{ik|x_s - x_t|}}{4\pi|x_s - x_t|} = \frac{e^{ik|d|}}{4\pi|d|} \lim_{L\to\infty} H_L(d,v) \simeq \frac{e^{ik|d|}}{4\pi|d|} e^{-ikv\cdot\hat{d}}$$

$$H_L(d,v) = \int_{S_2} T^L(d,\hat{x})\, e^{-ikv\cdot\hat{x}}\, d\sigma(\hat{x})$$

$$T^L(d,\hat{x}) = \frac{1}{4\pi} \sum_{\ell=0}^{L} \frac{i^\ell h_\ell^{(1)}(k|d|)}{h_0^{(1)}(k|d|)} (2\ell+1) P_\ell(\hat{x}\cdot\hat{d})$$

“Multipoles” Formulas II

$$eh^{\pm}(x_t) = \left[\frac{1}{k} \operatorname{curl} \operatorname{curl} \pm \operatorname{curl} \right] Sjm^{\pm}(x_t)$$

$$Sjm^{\pm}(x_t) = \int_{\Gamma} \frac{e^{ik|x_t - x_s|}}{4\pi |x_t - x_s|} jm^{\pm}(x_s)\, ds(x_s),$$

Γ centered at $c_s = 0$, $\operatorname{diam}\Gamma < 2|c_t| - 2|x_t - c_t|$

$$\Rightarrow \quad eh^{\pm}(x_t) = \lim_{L \to \infty} \frac{e^{ik|c_t|}}{4\pi |c_t|} H_L^{\pm}(c_t, x_t)$$

with $H_L^{\pm}(c_t, x_t) =$

$$\int_{S_2} T^L(c_t, \hat{x})\, eh_{\infty}{}^{\pm}(\hat{x})\, e^{-ik(x_t - c_t)\cdot \hat{x}}\, d\sigma(\hat{x})$$

$\Rightarrow$: 2-components FMM Algo

Diagona. of "Near to Far Fields Transform"

$eh^{\pm,\pm}(x_t, x_s, p_s)$: Right/Left Diffracted field by Γ at x_t

with Right/Left dipole located at x_s

${F_\infty}^{\pm,\pm}(\hat{u}_o; \hat{u}_i)$: Right/left component of the Far field at $\hat{u}_o$

Incident field : incidence $\hat{u}_i$

Right/left polarization: $\hat{t}^{\pm}(\hat{u}_i) = \dfrac{\hat{\theta}_i \pm i\hat{\varphi}_i}{\sqrt{2}}$

$$eh^{\pm,\pm}(x_t, x_s, p_s) = \frac{ik}{4\pi} \frac{e^{ik(|x_t|+|x_s|)}}{|x_t||x_s|} \lim_{L\to\infty} K_L^{\pm,\pm}(x_t, x_s, p_s)$$

$$K_L^{\pm,\pm}(x_t, x_s, p_s) =$$

$$\int_{S^2}\int_{S^2} T^L(x_t, \hat{u}_o) T^L(x_s, \hat{u}_i) p_s \cdot \hat{t}^{\pm}(\hat{u}_i) {F_\infty}^{\pm,\pm}(\hat{u}_o; -\hat{u}_i)$$

Correction of radar approximation

Assume $p_s.\hat{t}^{\pm}(\hat{x}_s) = 1$; remember that

$$T^L(d, \hat{x}) = \frac{1}{4\pi} \sum_{\ell=0}^{L} \frac{i^\ell h_\ell^{(1)}(k|d|)}{h_0^{(1)}(k|d|)} (2\ell+1) P_\ell(\hat{x} \cdot \hat{d})$$

and, when $k|d|$ large, L given,

$$\frac{i^\ell h_\ell^{(1)}(k|d|)}{h_0^{(1)}(k|d|)} \simeq 1 + \frac{1}{2ik|d|} \ell(\ell+1) + \ldots$$

$$\Rightarrow eh^{\pm,\pm}(x_t, x_s) \simeq \frac{ik}{4\pi} \frac{e^{ik(|x_t|+|x_s|)}}{|x_t||x_s|} K_{asym}^{\pm,\pm}(x_t, x_s)$$

$$\begin{aligned} K_{asym}^{\pm,\pm}(x_t, x_s) \sim\ & {F_\infty}^{\pm,\pm}(\hat{x}_t; -\hat{x}_s) \\ & + \frac{1}{2ik|x_s|} \Delta_{S_2,\hat{x}_s}\ {F_\infty}^{\pm,\pm}(\hat{x}_t; -\hat{x}_s) \\ & + \frac{1}{2ik|x_t|} \Delta_{S_2,\hat{x}_t}\ {F_\infty}^{\pm,\pm}(\hat{x}_t; -\hat{x}_s) \end{aligned}$$

Improvement at the beginning of the Franhöfer zone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