Self-adjointness of div h grad with a sign-changing h

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Based on a joint work with Claudio Cacciapuoti and Andrea Posilicano, Como
(to appear in Journal d'Analyse Mathématique)

Problem setting

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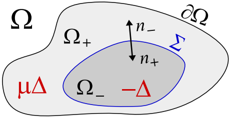

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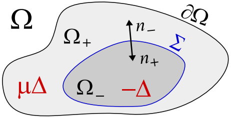

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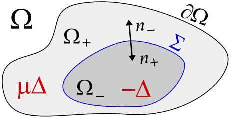

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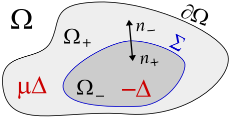

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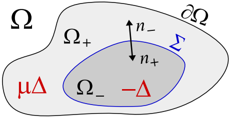

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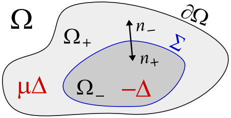

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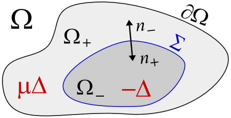

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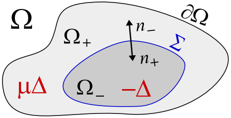

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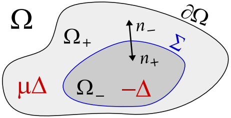

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Regularization: $L_\varepsilon = -\operatorname{div}(h + i\varepsilon)\operatorname{grad}$ is coercive for $\varepsilon > 0$, $L_\varepsilon u_\varepsilon = f$ is solvable, but $u_\varepsilon \xrightarrow{???} u$ as $\varepsilon \to 0$? (anomalously localized resonance: Ammari, Giraolo, Kang, Lee, Milton'13, Kohn, Lu, Schweizer, Weinstein'14, ..., H.-M. Ngyuên'15–).

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Properties of L?

Closedness, self-adjointness, compact resolvent, Fredholmness,...?

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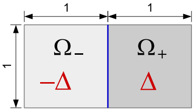

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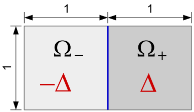

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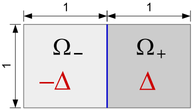

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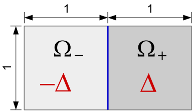

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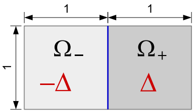

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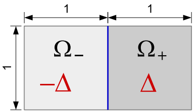

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Kostrykin, Krejčiřík, Makarov, Schmitz, ...'??

An abstract approach through indefinite quadratic forms (no exact announcement is available so far)

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In particular, $\mathscr{L}$ is not Fredholm.

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The operator L is essentially self-adjoint. If $\mathscr{L}$ is its (unique) self-adjoint extension, then

$$\mathscr{D}(\mathscr{L}) \supset \{u : \quad u_\pm \in H^1(\Omega_\pm), \quad \Delta u_\pm \in L^2(\Omega_\pm), \quad u_\pm \text{ satisfy (BC)}\}, \tag{$\star$}$$

$$u_- = u_+ \text{ and } \frac{\partial u_-}{\partial n_-} = \mu \frac{\partial u_+}{\partial n_+} \text{ on } \Sigma, \quad u_+ = 0 \text{ on } \partial\Omega \tag{BC}$$

The planar case $N = 2$

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The geometric hypotheses already appeared in a different context (well-posedness): Ola'95, Kettunen, Lassas, Ola'14, Vinoles'16

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For any $(\Pi, \Theta = \Theta^*)$ one has Krein's resolvent formula

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i.e. one has a reduction to a problem on the "boundary" $\mathscr{G}$.

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Boundary triple and Weyl function

Denote $\Lambda := \sqrt{1 - \Delta_{\partial\Omega}}$, then

$$\mathscr{G} = H^{\frac{1}{2}}(\partial\Omega), \quad \Gamma u = -\Lambda^{-1}\big(u|_{\partial\Omega}\big), \quad \Gamma' u = \frac{\partial\big(u - P_0(u|_{\partial\Omega})\big)}{\partial n}\Big|_{\partial\Omega}, \quad M_z = (D_0 - D_z)\Lambda,$$

with P_z and M_z being the Poisson operator and the Dirichlet-to-Neumann map,

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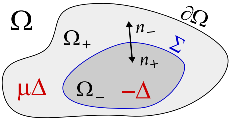

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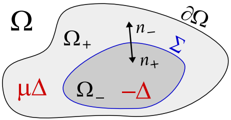

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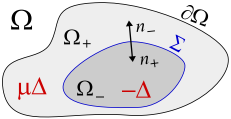

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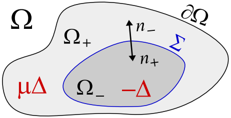

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$$\Pi\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix} = \frac12\begin{pmatrix} f_1+f_2\\ f_1+f_2\\ 0\end{pmatrix}, \quad \operatorname{ran}\Pi \simeq H^{\frac12}(\Sigma), \quad \Theta = \frac12(D_0^- - \mu D_0^+)\Lambda, \quad \mathscr{D}(\Theta) = H^{\frac52}(\Sigma).$$

Study of Θ

The operator Θ in $H^{\frac{1}{2}}(\Sigma)$ "controls" the properties of $L = A_{\Pi,\Theta}$.

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If each MCC(=maximal connected component) of Σ is strictly convex, then all principal curvatures are either all > 0 or all < 0 on MCC

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If each MCC(=maximal connected component) of Σ is strictly convex, then all principal curvatures are either all > 0 or all < 0 on MCC $\Rightarrow b_0$ does not vanish

$\mu = 1$ and $N \geq 3$

Strictly convex Σ

$$\Theta = \frac{1}{2}(D_0^- - D_0^+)\Lambda, \quad \mathscr{D}(\Theta) = H^{\frac{5}{2}}(\Sigma).$$

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Σ with a flat part

Explicit construction of a singular Weyl sequence implies $0 \in \sigma_{\mathrm{ess}}(\mathscr{L})$.

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- **Can $\sigma_{\mathrm{ess}}(\mathscr{L})$ be larger than $\{0\}$?**

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- **(Similarity to interior transmission eigenvalue problems?...)**