On global attractors of Hamilton nonlinear PDEs

Alexander Komech
Faculty of Mathemactics of Vienna University and
Dobrushin Lab., IITP RAS, Moscow

Waves diffracted by Patrick Joly

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alexander.komech@univie.ac.at

Abstract

Theory of attractors for Hamilton nonlinear PDEs was developed since 1995 together with P. Joly and H. Spohn and since 2006 together with A. Comech, V. Imaikin, E. Kopylova and B. Vainberg.

The problem is suggested by fundamental phenomena of classical and quantum physics: radiation damping in classical electrodynamics, Bohr's transitions to quantum stationary states, L. de Broglie's wave-particle duality, and other.

Plan

I. Physical motivations

II. Attraction to stationary states

III. Attraction to solitons

IV. Attraction to stationary orbits

V. Open problems

VI. General conjecture for G-invariant equations

VII. Comparison with attractors for dissipative systems

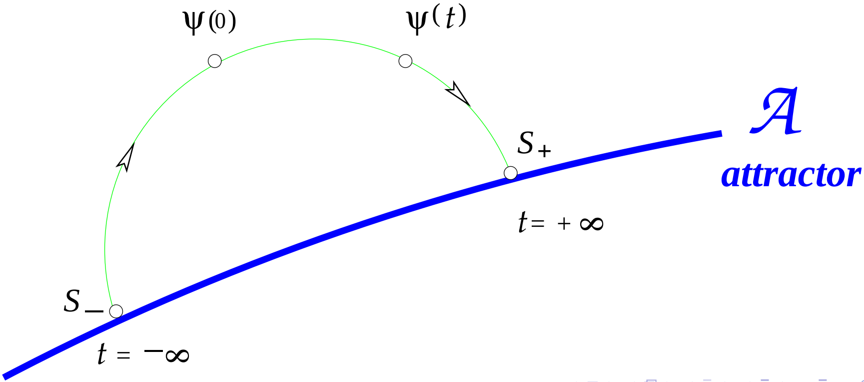
I. Physical motivations

N. Bohr: Quantum Transitions (1913)

(QT) $|E_-\rangle \mapsto |E_+\rangle$ Quantum Stationary Orbits

Dynamical interpretation: Long-time attraction

(A) $\psi(t) \sim S_\pm, \qquad t \to \pm\infty$



II. Attraction to stationary states

$$\ddot\psi(x,t)=\psi''(x,t)+f(x,\psi(x,t)),\quad x\in\mathbb{R}$$

$$f(x,\psi)=-\nabla_\psi U(x,\psi),\qquad x\in\mathbb{R},\ \psi\in\mathbb{R}^N$$

i) $U(x,\psi)\ge -C$, ii) $\min\limits_{x\in[a,b]} U(x,\psi)\to\infty$ as $|\psi|\to\infty$ (U)

iii) $U(x,\psi)=0$ for $|x|\ge c$.

The Hamilton functional

$$H=\frac12\int[|\dot\psi(x,t)|^2+|\psi'(x)|^2]dx+\int U(x,\psi(x))dx$$

The Hamilton form

$$\dot\psi(x,t)=\pi(x,t),\quad \dot\pi(x,t)=\psi''(x,t)+f(x,\psi(x,t)),\quad x\in\mathbb{R}$$

The set of stationary states $\mathcal{S} := \{\psi : \psi''(x) + f(x, \psi(x)) \equiv 0\}$

Theorem 1. (K. 1999) i) Let (U) hold. Then

$$\operatorname{dist}(\psi(\cdot, t), \mathcal{S}) \to 0, \qquad t \to \pm\infty. \qquad (A1)$$

where $\operatorname{dist}(\psi, \varphi) := \Sigma_{R=0}^{\infty} 2^{-R} \, \|\psi - \varphi\|_{H^1(-R,R)}$

ii) Let $f(x, \psi)$ be real analytic in $\psi \in \mathbb{R}^N$. Then the set S is discrete and

$$\psi(\cdot, t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} S_{\pm} \in \mathcal{S}, \qquad t \to \pm\infty. \qquad \text{(A2)}$$

Proof: Energy radiation to infinity.

Couterexample: (A2) can fail if the set $\mathcal{S}$ **is not discrete**. Let

$$f(x, \psi) \equiv 0 \quad \text{for} \quad x \in \mathbb{R}, \quad \psi \in [-1, 1]$$

Then (A2) fail for the solution $\psi(x, t) = \sin[\log(|x - t| + 2)]$

3D wave-particle system

$$\ddot{\psi}(x,t)=\Delta\psi(x,t)-\rho(x-q(t)),\quad x\in\mathbb{R}^3$$

$$\dot{q}=p/\sqrt{1+p^2},\qquad \dot{p}=-\nabla V(q)-\int\nabla\psi(x,t)\rho(x-q)\,dx$$

$\rho\in C_0^\infty(\mathbb{R}^3),\quad q\in\mathbb{R}^3.\qquad V(q)\to\infty$ as $|q|\to\infty$

Wiener condition (FGR): $\hat\rho(k):=\int e^{ikx}\rho(x)dx\neq 0,\ k\in\mathbb{R}^3 \qquad (W)$

$$H=\frac{1}{2}\int[|\pi(x)|^2+|\nabla\psi(x)|^2]\,dx+\int\psi(x)\rho(x-q)\,dx+\sqrt{1+p^2}+V(q)$$

Stationary states $\mathcal{S}:=\{(\psi,0,q,0):\Delta\psi(x)-\rho(x)\equiv 0,\nabla V(q)=0\}$.

Theorem 2. (K, Spohn, Kunze 1997) $Y(t)=(\psi(t),\pi(t),q(t),p(t))$

i) $\mathrm{dist}\,(Y(t),\mathcal{S})\to 0,\ \ t\to\pm\infty$.

ii) Let the set $\{q\in\mathbb{R}^3:\nabla V(q)=0\}$ be discrete in $\mathbb{R}^3$. Then $\psi(\cdot,t)\to S_\pm\in\mathcal{S},\qquad t\to\pm\infty$.

Proof: Energy radiation and Wiener Tauberian Theorem.

3D Maxwell-Lorentz Eqs. Abraham *extended electron*:

$$\dot E = \mathrm{rot}\ B - \dot q \rho(x-q),\ \ \dot B = -\mathrm{rot}\ E,\ \ \mathrm{div}\ E = \rho(x-q),\ \ \mathrm{div}\ B = 0$$

$$\dot q = \frac{p}{\sqrt{1+p^2}},\ \ \dot p = \int [E + E^{\rm ext} + \dot q(t) \wedge (B + B^{\rm ext})]\rho(x-q(t))\, dx$$

Scalar potential $E^{\rm ext}(x) = -\nabla\phi^{\rm ext}(x)$, $V(q) := \int \phi^{\rm ext}(x)\rho(x-q)\, dx$

Hamiltonian $H = \frac{1}{2}\int [E^2(x) + B^2(x)]\, dx + V(q) + \sqrt{1+p^2}$

Theorem 3. (K, H. Spohn 2000)

i) Let $V(q) \to \infty$ as $|q| \to \infty$, and (W) hold. Then

$\mathrm{dist}\,(Y(t), \mathcal{S}) \to 0,\ \ t \to \pm\infty.$

ii) Let the set $\{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$ be discrete in $\mathbb{R}^3$. Then the set $\mathcal{S}$ is discrete and $\psi(\cdot, t) \to S_\pm \in \mathcal{S}, \qquad t \to \pm\infty.$

III. Attraction to solitons

Translation-invariant wave-particle system: $V(q) \equiv 0$

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - \rho(x - q(t)), \quad x \in \mathbb{R}^3$$

$$\dot{q} = p/\sqrt{1+p^2}, \ \dot{p} = -\int \nabla\psi(x,t)\rho(x-q)\,dx$$

Solitons: $\psi(x,t) = \psi_v(x - vt - a)$.

Solitary manifold $\mathcal{S} := \{\psi_v(x-a) : |v| < 1, a \in \mathbb{R}^3\}$

Theorem 4. (K, H. Spohn 1998) Let (W) hold. Then

$$\psi(x,t) \sim \psi_{v_\pm}(x - q(t)), \quad q(t) \sim v_\pm t + \mathcal{O}(\log t),$$

$$\dot{q}(t) \to v_\pm, \qquad \ddot{q}(t) \to 0, \qquad t \to \pm\infty : \qquad \textit{Radiation Damping}$$

Extension to translation-invariant ML Eqs: K, V. Imaikin 2004

IV. Attraction to 'stationary orbits' $\psi(x)e^{i\omega t}$

1D KG Eqn coupled to nonlinear oscillator

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \delta(x)f(\psi(0,t)), \quad x \in \mathbb{R} \qquad (KG)$$

C1 $f(\psi) = -\nabla U(\psi), \qquad \inf U(\psi) > -\infty$

Hamiltonian $(\pi(x,t) := \dot\psi(x,t))$

$$H = \frac{1}{2}\int[|\pi(x)|^2 + |\psi'(x)|^2 + m^2|\psi(x)|^2]dx + U(\psi(0))$$

C2 $U(1)$**-invariance:** $U(\psi) = u(|\psi|)$

Solitary waves $\psi(x,t) = \psi_\omega(x)e^{i\omega t}$, $\omega \in \Omega \subset (-m, m)$

Nonlinear eigenvalue problem

$$-\omega^2\psi_\omega(x) = \psi_\omega''(x) - m^2\psi_\omega(x) + \delta(x)f(\psi_\omega(0))$$

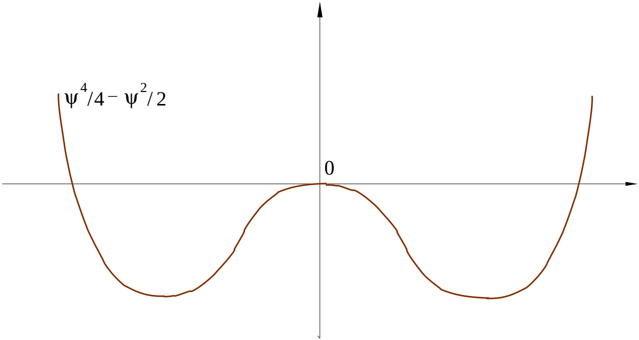
$$\psi_\omega(x) = Ce^{-\varkappa|x|}, \qquad 2\varkappa C = f(C), \qquad \varkappa := \sqrt{m^2 - \omega^2} > 0$$

Solitary manifold $\mathcal{S} = \{e^{i\theta}\psi_\omega(x) : \omega \in \Omega, \ \theta \in [0, 2\pi]\}$.

C3 Equation (KG) is strictly nonlinear:

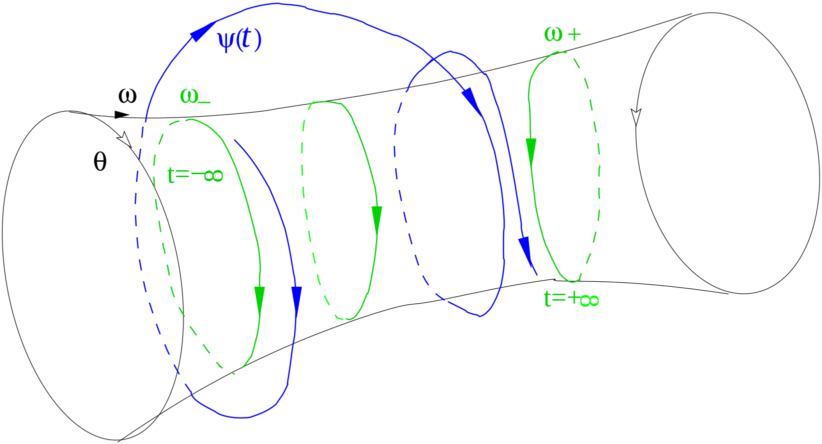
$$U(\psi) = u(|\psi|^2) = \Sigma_0^N u_j |\psi|^{2j}, \quad u_N > 0, \quad N \geq 2 \qquad \text{(SNL)}$$

Example: *Ginzburg-Landau potential* $U(\psi) = \psi^4/4 - \psi^2/2$,
$f(\psi) = -|\psi|^2\psi + \psi$



Theorem 5. (K. 2003) Let C1 – C3 hold. Then for any solution $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$

$$\psi(\cdot, t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \mathcal{S}, \qquad t \to \pm\infty \qquad (A3)$$



Proof

i) *Omega-limit trajectories* $\beta(x,t) = \lim_{s_k\to\infty} \psi(x, s_k + t)$

Example: $\psi(x,t) \sim \psi_*(x)e^{i\omega_* t} \Longrightarrow \beta(x,t) = e^{i\theta}\psi_*(x)e^{i\omega_* t}$

Theorem 5 $\Longleftrightarrow$ Each omega-limit trajectory $\beta(x,t) \equiv \psi_*(x)e^{i\omega_* t}$

Equivalently, $\tilde\beta(x,\omega) = \delta(\omega - \omega_*)\psi_*(x)$

ii) *Equation:* $\ddot\beta(x,t) = \beta''(x,t) - m^2\psi + \delta(x)f(\beta(0,t)), \quad x, t \in \mathbb{R}$

iii) *Fourier transform in time*: $-\omega^2\tilde\beta(x,\omega) = \tilde\beta''(x,\omega) - m^2\psi + \delta(x)\tilde f(\omega)$

iv) **Tithcmarsh Convolution Theorem (1926)**

Generalizations: The attraction (A1)–(A3) is proved

i) in 2006-2011 together with A. Comech for Eqns

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \sum_{k=1}^{N}\delta(x-x_k)f_k(\psi(x_k,t)), \quad x\in\mathbb{R}$$

$$\ddot\psi(x,t) = \Delta\psi(x,t) - m^2\psi + \rho(x)f(\langle\psi(\cdot,t),\rho\rangle), \qquad x\in\mathbb{R}^n$$

$$i\dot\psi(x,t) = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\psi + \rho(x)f(\langle\psi(\cdot,t),\rho\rangle), \qquad x\in\mathbb{R}^n$$

The Wiener condition (Fermi Golden Rule) : $\quad \hat\rho(k)\neq 0,\ k\in\mathbb{R}^n$

ii) in 2012 by A. Comech for discrete in space and time nonlinear KG.

iii) in 2016-2017 by E. Kopylova for 3D nonlinear wave and KG Eqns with concentrated nonlinearities.

V. Open problems: the Attraction (A1)–(A3) for

i) the Klein-Gordon equation (KG) and *fixed solitary waves*

$\psi(x,t) \sim \psi_{\omega_\pm}(x) e^{i\omega_\pm t}, \quad t \to \pm\infty$

ii) the Schrödinger equation $\quad i\dot\psi(x,t) = \psi''(x,t) + \delta(x) f(\psi(0,t))$

iii) **relativistic** nonlinear KG Eqn

$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t) + f(\psi(x,t)), \quad f(\psi) = -\nabla_\psi U(\psi)$

iv) coupled Maxwell-Schrödinger equations

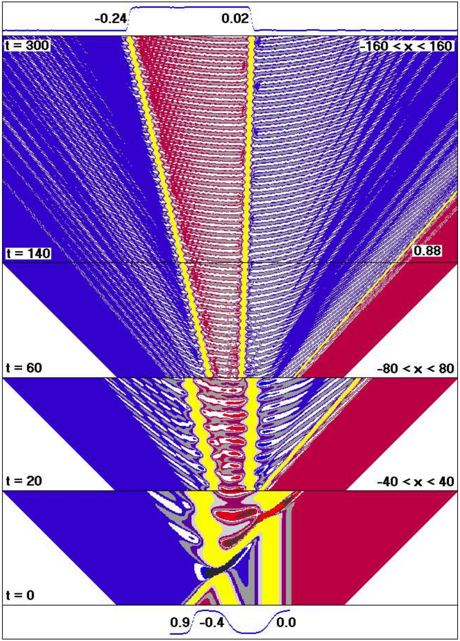
$i\dot\psi(x,t) = \frac{1}{2}[-i\nabla + \mathbf{A}(x,t) + \mathbf{A}^{\rm ext}(x,t)]^2\psi + [A_0(x,t) + A_0^{\rm ext}(x)]\psi$

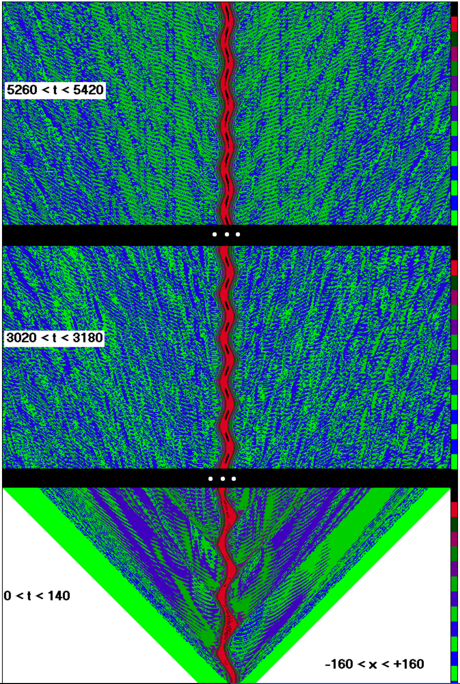
$\Box A_\alpha(x,t) = 4\pi J_\alpha(x,t), \qquad \alpha = 0,1,2,3$

v) coupled Maxwell-Dirac equations

$\sum_{\alpha=0}^{3} \gamma^\alpha [i\nabla_\alpha - A_\alpha(x,t) - A_\alpha^{\rm ext}(x,t)]\psi(x,t) = m\psi(x,t)$

$\Box\, A_\alpha(x,t) = J_\alpha(x,t) := \overline{\psi(x,t)}\gamma^0\gamma_\alpha\psi(x,t), \quad \alpha = 0,1,2,3$


-0.24
0.02
t = 300
-160 < x < 160
t = 140
0.88
t = 60
-80 < x < 80
t = 20
-40 < x < 40
t = 0
0.9
-0.4
0.0


5260 < t < 5420
3020 < t < 3180
0 < t < 140
-160 < x < +160

VI. General conjecture for G-invariant equations

The results on the attraction correspond to the symmetry group of Eqns:

I. Attraction to stationary states: $G = e$

II. Attraction to solitons: $G = \mathbb{R}^n$

III. Attraction to stationary orbits: $G = U(1)$

G-Conjecture: For 'generic' G-invariant Hamilton nonlinear PDEs

$$\psi(t) \sim e^{g_\pm t}\psi_\pm, \qquad t \to \pm\infty$$

for each finite energy solution, where $g_\pm \in \mathfrak{g}$.

Experimental confirmation **symmetry group $\leftrightarrow$ asymptotic states**

Gell-Mann, Ne'eman 1961: Dynkin scheme of $su(3)$ with 8 generators corresponds to 8 baryons ('eightfold way').

7 baryons were known in 1961, 8-th Ω^--hyperon was discovered in 1964.

VII. Comparison with attractors for dissipative PDEs

For dissipative systems 1970–

Navier-Stokes eqns, reaction-diffusion eqns, nonlinear parabolic eqns, damped wave eqns, etc

i) The convergence holds in **bounded and unbounded regions**;

ii) In **global energy norm**;

iii) For $t \to +\infty$ **only**.

For Hamiltonian PDEs

i) The convergence holds **only** in unbounded regions;

ii) In **local energy seminorms**;

iii) For $t \to \pm\infty$.

Happy Birthday, Patrick!

THANK YOU !