How well can the Helmholtz kernel be compressed?

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Joint work with Sergey Solovyev
Motivation: Recent Iterative Solvers

- Sweeping Preconditioner (Engquist and Ying 2011)
- Single Layer Potential Method (Stolk 2013)
- Source Transfer (Chen and Xiang 2013)
- Method of Polarized Traces (Zepeda-Núñez and Demanet 2015)
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Based on the block $LU$ factorization $A = LU$ with factors

\[
\begin{pmatrix}
T_1 & L_1 & T_2 & \cdots & \cdots & L_{J-2} & T_{J-1} & L_{J-1} & T_J \\
L_1 & T_2 & \cdots & \cdots & \cdots & L_{J-2} & T_{J-1} & L_{J-1} & T_J
\end{pmatrix}
\begin{pmatrix}
I_1 & T_1^{-1} U_1 & I_2 & T_2^{-1} U_2 & \cdots & \cdots & I_{J-1} & T_{J-1}^{-1} U_{J-1} & I_J
\end{pmatrix}
\]
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Based on the block $LU$ factorization $A = LU$ with factors

$$
\begin{bmatrix}
T_1 & & & \\
L_1 & T_2 & & \\
& \ddots & \ddots & \\
L_{J-2} & T_{J-1} & & \\
& & L_{J-1} & T_J
\end{bmatrix}
\begin{bmatrix}
I_1 & T_1^{-1}U_1 & & \\
& I_2 & T_2^{-1}U_2 & \\
& & \ddots & \ddots \\
& & I_{J-1} & T_{J-1}^{-1}U_{J-1}
\end{bmatrix}
$$

At the continuous level, this corresponds to optimal Schwarz methods (Nataf and Rogier 1994, Després 1991, Lions 1990, Hagstrom et al 1988)

Rigorous equivalence proofs in G and Zhang (2017)
Helmholtz Model Problem

\[(\Delta + k^2)u = f \quad \text{in } \Omega := (0, 1)^2,\]
\[Bu = 0 \quad \text{on } \partial \Omega,\]

\(B\) denotes a suitable boundary operator. Standard centered finite difference discretization leads to the linear system

\[Au = f.\]

Isolating the first column of unknowns leads to the partitioned system

\[
\begin{pmatrix}
A_1 & A_{12} \\
A_{21} & A_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}.
\]

The Helmholtz kernel is contained in the Schur complement

\[S := A_1 - A_{12}A_2^{-1}A_{21} \quad Su_1 = f_1 - A_{12}A_2^{-1}f_2.\]
Definition: The \( \epsilon \)-rank of a matrix \( A \) is the smallest integer \( r_\epsilon(A) \) such that \( \frac{\sigma_j(A)}{\sigma_1(A)} < \epsilon \) for all \( j > r_\epsilon(A) \).
Laplace and Helmholtz Large Subblock of $S$

For $S$ of size $n = 64$ large subblock $S_{m,k}$ for $m = 31$:

- Laplace Schur complement subblock decays rapidly
- Helmholtz Schur complement subblock oscillates
- $S = U\Sigma V^T$ corresponds to $G(y, \eta) = \sum_i u_i(y)v_i(\eta)$ at the continuous level
- Separability of $G(y, \eta)$ related to decay of $\sigma_i$
Laplace and Helmholtz Large Subblock of $S$

For $S$ of size $n = 128$ large subblock $S_{m,k}$ for $m = 63$:

- Laplace, $k = 0$
- Helmholtz, $k = 80.4$

- Laplace Schur complement subblock decays rapidly
- Helmholtz Schur complement subblock oscillates
- $S = UΣV^T$ corresponds to $G(y, η) = \sum_i u_i(y)v_i(η)$ at the continuous level
- Separability of $G(y, η)$ related to decay of $σ_i$
Laplace and Helmholtz Singular Value Comparison

![Graph comparing singular values of Laplace and Helmholtz kernels](image)

- **Laplace**
- **Helmholtz**

**Axes:**
- **x-axis (j)**: Index
- **y-axis**: Logarithmic scale (10 to 10^5)

**Data Points:**
- Laplace: Red circles
- Helmholtz: Blue stars

**Legend:**
- Laplace
- Helmholtz

**Explanation:** This graph compares the singular values of Laplace and Helmholtz kernels for various indices (j). The logarithmic scale on the y-axis helps visualize the decay of singular values effectively.
Laplace and Helmholtz Singular Value Comparison

![Graph showing the comparison of Laplace and Helmholtz singular values.](image-url)
Laplace and Helmholtz Singular Value Comparison

![Graph comparing singular values of Laplace and Helmholtz kernels.](image)

**Introduction**

**Motivation**

**Model Problem**

**H-Matrices**

**Literature**

- Martinsson Rokhlin
- Banjai Hackbusch
- Engquist Zhao
- Börm Melenk
- Betcke et al
- Delamotte

**Numerics**

- 2d One Side
- 2d More Sides
- 3d One Face
- 3d Opposite Faces

**Conclusions**
Martinsson, Rokhlin (2007)

A Fast Direct Solver for Scattering Problems Involving Elongated Structures

“Most such schemes rely on rank deficiencies in the off-diagonal blocks of the matrix, which means they do not perform asymptotically fast for high-frequency problems (not $O(N \log^q N)$) [...]"

To the author’s best knowledge, there does not exist a 'fast’ algorithm for the direct solution of high frequency problems on general domains.

However, for the special case of scattering from elongated objects [...] the system can be solved directly in $O(N \log^2 N)$ operations.

⇒ polylogarithmic-linear complexity for a true solver!
"[The Theorem] states that to within precision $\epsilon$, the rank of interaction between the two boxes in Fig. 1, of lengths $L$ and fixed widths, is $O(\log(kL)\log \epsilon^2)$, as $L \to \infty$ and $\epsilon \to 0$ [provided that $S \geq c|\log \epsilon|/k$. [...] The Theorem does not specify how the interaction rank depends on the width $H$ of the boxes. In fact, it grows quite rapidly with the width."

Table 1. The table displays the number of interpolation points $J$ actually required for the interpolation (3.12) for a separation $S$ such that $kS = 1$. 

<table>
<thead>
<tr>
<th>$kL$</th>
<th>$\epsilon = 10^{-5}$</th>
<th>$\epsilon = 10^{-10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$kH = 1$</td>
<td>$10^1$</td>
<td>$10^2$</td>
</tr>
<tr>
<td>$kH = 2$</td>
<td>$15$</td>
<td>$20$</td>
</tr>
<tr>
<td>$kH = 3$</td>
<td>$20$</td>
<td>$25$</td>
</tr>
<tr>
<td>$kH = 4$</td>
<td>$24$</td>
<td>$28$</td>
</tr>
</tbody>
</table>
Banjai and Hackbusch (2008)

Hierarchical matrix techniques for low and high frequency Helmholtz problems

“The high frequency problem presents a considerably more difficult challenge [...] but recently multilevel implementations were reported on, both in 2D and 3D, with complexity $O(n \log^a n)$ for some small constant $a$ (see [Amini and Profit 2003, Darve 2000, Lu and Chew 1994]).

Unfortunately, [...] numerical instability problems render the approximation unusable.

We detect the blocks for which the approximation is unstable, and approximate these blocks by an H-matrix which can be computed in a stable manner without the use of the multipole expansion. It is possible to do this efficiently, since these blocks stem from the discretization of parts of the boundary that are small compared to the wavelength.”

$\implies O(n \log^2 n)$ for construction and matrix vector product!
Engquist and Zhao (2014)

“X and Y are two disjoint compact surfaces in 3D, $\dim(X) = \dim(Y) = 2$. Then for $k$ large enough

$$k^{2-\delta} \lesssim r_\epsilon \lesssim k^{2+\delta}$$

“For a discretization with a fixed ratio between the grid size and the wavelength, the sharp lower bound means that the off-diagonal sub-matrices of the linear system after elimination of the interior nodes is of full rank modulo a constant in the high frequency limit.”

$\implies$ non-compressibility of the Helmholtz Kernel for constant patches!
Börm and Melenk (2016)

Approximation of the high-frequency Helmholtz kernel by nested directional interpolation

“In many of these approaches, the underlying justification is based on approximating the kernel using suitable expansion systems; in the high frequency setting, these expansion systems are typically not polynomial in order to account for the oscillatory behavior of $k$.

Polylogarithmic-linear complexity of the algorithms requires a second ingredient: a multilevel structure. Suitable expansion systems are given on each level and a fast transfer between levels has to be effected, e.g., by re-expansion”.

Polylogarithmic-linear compression of the Schur complement matrix!
Betcke, van’t Wout and Gélat (2016)

Computationally efficient boundary element methods for high-frequency Helmholtz problems in unbounded domains

“While the $H$-matrix based discretization described in this chapter performs well for many Helmholtz problems, a direct improvement is possible by moving towards $H^2$-matrix techniques. They allow for a considerable memory reduction, but similar to $H$-matrices, they are not asymptotically optimal at high frequencies.”

(see also Lizé PhD thesis 2014 and Gatto, Hesthaven 2016)
La fréquence de 2GHz correspond à la fréquence maximale que l’on peut utiliser sur ce maillage contenant N = 193707 degrés de liberté. Sur la figure 4.22, le bloc dans le coin supérieur droit présente une croissance linéaire en la fréquence malgré une section efficace maximale.
PhD thesis by Delamotte (2016)

Une étude du rang du noyau de l’équation de Helmholtz: application des H-matrices à l’EFIE

Figure 4.22 – Représentation graphique du rang des blocs de l’approximation $\mathcal{H}$-matrice de la matrice de l’EFIE sur l’ellipsoïde pour des fréquences différentes.

La fréquence de 2GHz correspond à la fréquence maximale que l’on peut utiliser sur ce maillage contenant $N = 193707$ degrés de liberté. Sur la figure 4.22, le bloc dans le coin supérieur droit présente une croissance linéaire en la fréquence malgré une section efficace maximale.

Gene Golub: “I want to see numbers!”
<table>
<thead>
<tr>
<th>$\epsilon = 1e$</th>
<th>Robin</th>
<th>Wave guide</th>
<th>Dirichlet</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>Large Schur complement block with $m = \frac{n}{2} - 1$</td>
<td>Medium Schur complement block with $m = \frac{n}{4} - 1$</td>
<td>Small Schur complement block with $m = \frac{n}{8} - 1$</td>
</tr>
<tr>
<td>20.1</td>
<td>5 6 7 7 8</td>
<td>6 6 7 7 8</td>
<td>4 (3) 5 (4) 6 (5) 6 (5) 7 (6)</td>
</tr>
<tr>
<td>40.2</td>
<td>8 10 10 11 12</td>
<td>9 10 10 11 11</td>
<td>7 (4) 9 (5) 10 (6) 10 (6) 11 (7)</td>
</tr>
<tr>
<td>80.4</td>
<td>12 14 16 18 18</td>
<td>14 15 17 18 18</td>
<td>11 (4) 14 (5) 15 (6) 17 (7) 18 (8)</td>
</tr>
<tr>
<td>160.8</td>
<td>18 22 24 28 30</td>
<td>23 26 27 29 30</td>
<td>19 (4) 23 (6) 25 (7) 27 (8) 29 (9)</td>
</tr>
<tr>
<td>321.6</td>
<td>29 34 40 44 47</td>
<td>38 41 44 46 49</td>
<td>30 (4) 36 (6) 41 (7) 43 (9) 46 (10)</td>
</tr>
<tr>
<td>643.2</td>
<td>50 59 67 73 79</td>
<td>67 72 76 80 83</td>
<td>46 (4) 61 (6) 68 (8) 72 (9) 76 (11)</td>
</tr>
<tr>
<td>20.1</td>
<td>3 4 4 4 5 5</td>
<td>3 4 4 4 5</td>
<td>3 (2) 3 (3) 3 (3) 5 (3) 5 (4)</td>
</tr>
<tr>
<td>40.2</td>
<td>5 5 6 6 7 7</td>
<td>5 6 6 7 7</td>
<td>4 (2) 5 (3) 6 (3) 6 (3) 7 (4)</td>
</tr>
<tr>
<td>80.4</td>
<td>6 8 9 9 10 10</td>
<td>8 9 10 10 10</td>
<td>7 (2) 8 (3) 8 (3) 10 (3) 10 (4)</td>
</tr>
<tr>
<td>160.8</td>
<td>9 12 14 16 17</td>
<td>13 15 16 16 17</td>
<td>11 (2) 14 (3) 16 (3) 16 (4) 17 (4)</td>
</tr>
<tr>
<td>321.6</td>
<td>15 18 22 24 27</td>
<td>22 24 28 28 29</td>
<td>21 (2) 23 (3) 25 (3) 28 (4) 29 (4)</td>
</tr>
<tr>
<td>643.2</td>
<td>25 32 38 42 45</td>
<td>42 45 48 49 51</td>
<td>36 (2) 43 (3) 45 (3) 49 (4) 49 (4)</td>
</tr>
</tbody>
</table>

Compressibility of Helmholtz Kernels

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$\epsilon$-rank

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Conclusions
10 ppw: all $\epsilon$-ranks of $S_{m,k}$ grow in 2D like $k^{3/4}$!
Including more sides in Schur complement
Including more sides in Schur complement

As example for the case of Dirichlet boundary conditions

<table>
<thead>
<tr>
<th>$\epsilon = 1e$</th>
<th>Two adjacent sides</th>
<th>Two opposite sides</th>
<th>Three sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>Large Schur</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>complement block</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = \frac{n}{2} - 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.10</td>
<td>9 12 13 13 14</td>
<td>7 7 7 8 8</td>
<td>11 11 12 13 13</td>
</tr>
<tr>
<td>31.92</td>
<td>14 15 17 20 21</td>
<td>13 14 14 14 14</td>
<td>17 18 19 20 20</td>
</tr>
<tr>
<td>50.66</td>
<td>21 23 26 28 30</td>
<td>26 27 27 27 27</td>
<td>30 31 32 33 33</td>
</tr>
<tr>
<td>80.42</td>
<td>38 41 44 46 49</td>
<td>52 53 53 53 53</td>
<td>36 57 58 58 59</td>
</tr>
<tr>
<td>127.67</td>
<td>68 72 74 78 81</td>
<td>104 105 105 105 105</td>
<td>108 109 110 111 112</td>
</tr>
<tr>
<td>202.66</td>
<td>130 134 138 142 145</td>
<td>123 208 209 209 209</td>
<td>212 213 214 215 217</td>
</tr>
</tbody>
</table>

With 10ppw and including opposite sides in $S$, $\epsilon$-ranks are $O(k)$
Schur complement for one face in 3D with 10ppw

As example for the case of Dirichlet boundary conditions

<table>
<thead>
<tr>
<th>10ppw</th>
<th>Schur complement for one side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon = 1 \times 10^{-e}$</td>
<td>-2 -3 -4 -5 -6</td>
</tr>
<tr>
<td>$k$</td>
<td>Small block</td>
</tr>
<tr>
<td>5</td>
<td>1 2 3 5 6</td>
</tr>
<tr>
<td>10</td>
<td>3 4 4 7 8</td>
</tr>
<tr>
<td>20</td>
<td>6 9 12 12 14</td>
</tr>
<tr>
<td>40</td>
<td>15 23 27 32 39</td>
</tr>
</tbody>
</table>

With one face, $\epsilon$-ranks of $S_{m,k}$ grow in 3D like $k^{\frac{4}{3}}$!
Two opposite faces in 3D with 10ppw

As example for the case of Dirichlet boundary conditions

<table>
<thead>
<tr>
<th>10ppw</th>
<th>Schur complement for two opposite sides</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Small block</td>
</tr>
<tr>
<td>$\epsilon = 1e$</td>
<td>-2 -3 -4 -5 -6</td>
</tr>
<tr>
<td>$k$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2 3 5 8 10</td>
</tr>
<tr>
<td>10</td>
<td>3 4 6 7 8</td>
</tr>
<tr>
<td>20</td>
<td>9 11 14 17</td>
</tr>
<tr>
<td>40</td>
<td>26 35 43 51</td>
</tr>
</tbody>
</table>

With opposite faces, $\epsilon$-ranks of $S_{m,k}$ grow in 3D like $k^2$!
Conclusions

- The $\epsilon$-rank for off diagonal blocks is
  - constant for the Laplace equation
  - growing algebraically in $k$ for the Helmholtz equation

- In 2D, the growth measured is
  - $k^{\frac{3}{4}}$ for Schur complements on one side
  - $k$ for Schur complements including opposite sides

- In 3D, the growth measured is
  - $k^{\frac{4}{3}}$ for Schur complements on one face
  - $k^2$ for Schur complements including opposite faces

- Proving these results is currently work in progress

Preprint available at www.unige.ch/~gander/